Math. Z. (2007) 256:737–748
DOI 10.1007/s00209-006-0097-9

Mathematische Zeitschrift

Cohomology and extension problems for semi q-coronae

Alberto Saracco · Giuseppe Tomassini

Received: 1 February 2005 / Accepted: 21 September 2005 /
Published online: 6 January 2007

Abstract We prove some extension theorems for analytic objects, in particular sections of a coherent sheaf, defined in semi q-coronae of a complex space. Semi q-coronae are domains whose boundary is the union of a Levi flat part, a q-pseudoconvex part and a q-pseudoconcave part. Such results are obtained mainly using cohomological techniques.

Keywords q-pseudoconvexity · Cohomology · Extension

Mathematics Subject Classification Primary 32D15; Secondary 32F10 · 32L10

1 Introduction

Let X be a (connected and reduced) complex space. We recall that X is said to be *strongly q-pseudoconvex* in the sense of Andreotti and Grauert [1] if there exists a compact subset K and a smooth function $\varphi : X \to \mathbb{R}$, $\varphi \geq 0$, which is strongly q-plurisubharmonic on $X \smallsetminus K$ and such that:

(a) $0 = \min\limits_{X} \varphi < \min\limits_{K} \varphi$;
(b) for every $c > \max\limits_{K} \varphi$ the subset

$$B_c = \{x \in X : \varphi(x) < c\}$$

is relatively compact in X.

Supported by the MURST project "Geometric Properties of Real and Complex Manifolds".

A. Saracco · G. Tomassini (✉)
Scuola Normale Superiore, Piazza dei Cavalieri 7, Pisa 56126, Italy
e-mail: g.tomassini@sns.it

A. Saracco
e-mail: a.saracco@sns.it

If $K = \varnothing$, X is said to be *q-complete*. We remark that, for a space being 1-complete is equivalent to being Stein.

Replacing the condition (b) by

(b') for every $0 < \varepsilon < \min_K \varphi$ and $c > \max_K \varphi$ the subset

$$B_{\varepsilon,c} = \{x \in X : \varepsilon < \varphi(x) < c\}$$

is relatively compact in X,

we obtain the notion of *q-corona* (see [1,2]).

A q-corona is said to be *complete* whenever $K = \varnothing$.

The extension problem for analytic objects defined on q-coronae was studied by many authors (see e.g. [7,15–17,19]). In this paper we deal with the larger class of the semi q-coronae which are defined as follows. Consider a strongly q-pseudoconvex space (or, more generally, a q-corona) X, and a smooth function $\varphi : X \to \mathbb{R}$ displaying the q-pseudoconvexity of X. Let $B_{\varepsilon,c} \subset X$ and let $h : X \to \mathbb{R}$ be a pluriharmonic function (i.e. locally the real part of a holomorphic function) such that $K \cap \{h = 0\} = \varnothing$. A connected component of $B_{\varepsilon,c} \smallsetminus \{h = 0\}$ is, by definition, a *semi q-corona*.

Another type of semi q-corona is obtained by replacing the zero set of h with the intersection of X with a Levi flat hypersurface. More precisely, consider a closed strongly q-pseudoconvex subspace X of an open subset of $\mathbb{C}^n$ and the q-corona $C = B_{\varepsilon,c} = B_c \smallsetminus \overline{B}_\varepsilon$. Let H be a Levi flat hypersurface of a neighbourhood U of $\overline{B}_c$ such that $H \cap K = \varnothing$. The connected components C_m of $C \smallsetminus H$ are called semi q-coronae.

In both cases the semi q-coronae are differences $A_c \smallsetminus \overline{A}_\varepsilon$ where A_c, A_ε are strongly q-pseudoconvex spaces. Indeed, the function $\psi = -\log h^2$ (respectively $\psi = -\log \delta_H(z)$, where $\delta_H(z)$ is the distance of z from H) is plurisubharmonic in $W \smallsetminus \{h = 0\}$ (respectively $W \smallsetminus H$) where W is a neighbourhood of $B_c \cap \{h = 0\}$ (respectively $B_c \cap H$). Let $\chi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function such that $\chi \circ \varphi > \psi$ on a neighbourhood of $B_c \smallsetminus W$. The function $\Phi = \sup(\chi \circ \varphi, \psi) + \varphi$ is an exhaustion function for $B_c \smallsetminus \{h = 0\}$ (for $B_c \smallsetminus H$) and it is strongly q-plurisubharmonic in $B_c \smallsetminus (\{h = 0\} \cup K)$ (in $B_c \smallsetminus H \cup K$).

The interest for domains whose boundary contains a "Levi flat part" originated from an extension theorem for CR-functions proved in [13] (see also [10,11,18]).

Using cohomological techniques developped in [1–4] we prove that, under appropriate regularity conditions, holomorphic functions defined on a complete semi 1-corona "fill in the holes" (Corollaries 4 and 6). Meanwhile we also obtain more general extension theorems for sections of coherent sheaves (Theorems 3 and 5). As an application, we finally obtain an extension theorem for divisors (Theorems 15 and 19) and for analytic sets of codimension one (Theorem 17).

We remark that this approach fails in the case when the objects to be extended are not sections of a sheaf defined on the whole B_c. In particular, this applies for analytic sets of higher codimension. This is closely related with the general, definitely more difficult, problem of extending analytic objects assigned on some semi q-corona when the subsets B_c are not relatively compact in X i.e. when X is a genuine q-corona. It is worth noticing that a similar extension theorem for complex submanifold of higher codimension has been recently obtained in [5] by different methods based on Harvey-Lawson's theorem [8].

We wish to thank Mauro Nacinovich and Viorel Vâjâitu for their kind help and suggestions.

2 Cohomology and extension of sections

2.1 Closed q-coronae

Let X be a strictly q-pseudoconvex space (respectively $X \subset \mathbb{C}^n$ be a strictly q-pseudoconvex open set) and $H = \{h = 0\}$ (respectively H Levi Flat), and $C = B_{\varepsilon,c} = B_c \smallsetminus \overline{B}_\varepsilon$ a q-corona.

We can suppose that $B_c \smallsetminus H$ has two connected components, B_+ and B_-, and define $C_+ = B_+ \cap C$, $C_- = B_- \cap C$.

If $\mathcal{F} \in \mathrm{Coh}(B_c)$, we define $p(\mathcal{F}) = \inf\limits_{x \in B_c} \mathrm{depth}(\mathcal{F}_x)$, the depth of $\mathcal{F}$ on B_c. If $\mathcal{F} = \mathcal{O}$, the structure sheaf of X, we define $p(B_c) = p(\mathcal{O})$.

Theorem 1 *Let $\mathcal{F} \in \mathrm{Coh}(B_c)$. Then the image of the homomorphism*

$$H^r(\overline{B}_+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}_+, \mathcal{F})$$

(all the closures are taken in B_c), defined by $(\xi \oplus \eta) \mapsto \xi_{|\overline{C}_+} - \eta_{|\overline{C}_+}$ has finite codimension provided that $q - 1 \le r \le p(\mathcal{F}) - q - 2$.

Proof Consider the Mayer-Vietoris sequence applied to the closed sets $\overline{B}_+$ and $\overline{C}$

$$\begin{aligned} \cdots \to H^r(\overline{B}_+ \cup \overline{C}, \mathcal{F}) \to H^r(\overline{B}_+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \overset{\delta}{\to} \\ \overset{\delta}{\to} H^r(\overline{C}_+, \mathcal{F}) \to H^{r+1}(\overline{B}_+ \cup \overline{C}, \mathcal{F}) \to \cdots \end{aligned} \tag{1}$$

$\delta(a \oplus b) = a_{|\overline{C}_+} - b_{|\overline{C}_+}$. $\overline{B}_+ \cup \overline{C} = B_c \smallsetminus U$ where $U = B_- \cap B_\varepsilon$. U is q-complete, so the groups of compact support cohomology $H^r_c(U, \mathcal{F})$ are zero for $q \le r \le p(\mathcal{F}) - q$.

From the exact sequence of compact supports cohomology

$$\begin{aligned} \cdots \to H^r_c(U, \mathcal{F}) \to H^r(B_c, \mathcal{F}) \to \\ \to H^r(B_c \smallsetminus U, \mathcal{F}) \to H^{r+1}_c(U, \mathcal{F}) \to \cdots \end{aligned} \tag{2}$$

it follows that

$$H^r(B_c, \mathcal{F}) \overset{\sim}{\to} H^r(B_c \smallsetminus U, \mathcal{F}), \tag{3}$$

for $q \le r \le p(\mathcal{F}) - q - 1$.

Since B_c is q-pseudoconvex,

$$\dim_{\mathbb{C}} H^r(B_c, \mathcal{F}) < \infty$$

for $q \le r$ [1, Théorème 11], and so

$$\dim_{\mathbb{C}} H^r(B_c \smallsetminus U, \mathcal{F}) < \infty$$

for $q \le r \le p(\mathcal{F}) - q - 1$.

From (1) we see that $\dim_{\mathbb{C}} H^r(B_c \smallsetminus U, \mathcal{F}) = \dim_{\mathbb{C}} H^r(\overline{B}_+ \cup \overline{C}, \mathcal{F})$ is greater than or equal to the codimension of the image of the homomorphism δ. $\square$

Corollary 2 *Under the same assumption of Theorem 1, if $K \cap H = \varnothing$,*

$$\dim_{\mathbb{C}} H^r(\overline{C}_+, \mathcal{F}) < \infty$$

for $q \le r \le p(\mathcal{F}) - q - 2$.

Proof Since $K \cap H = \varnothing$, $\overline{B}_+$ is a q-pseudoconvex space, and by virtue of [1, Théorème 11] we have

$$\dim_{\mathbb{C}} H^r(\overline{B}_+, \mathcal{F}) < \infty$$

for $r \geq q$. On the other hand, $\overline{C}$ is a q-corona, thus we obtain

$$\dim_{\mathbb{C}} H^r(\overline{C}, \mathcal{F}) < \infty$$

for $q \leq r \leq p(\mathcal{F}) - q - 1$ in view of [2, Theorem 3]. By Theorem 1 we then get that for $q \leq r \leq p(\mathcal{F}) - q - 1$ the vector space $H^r(\overline{B}_+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F})$ has finite dimension and for $q-1 \leq r \leq p(\mathcal{F}) - q - 2$ its image in $H^r(\overline{C}_+, \mathcal{F})$ has finite codimension. Thus $H^r(\overline{C}_+, \mathcal{F})$ has finite dimension for $q \leq r \leq p(\mathcal{F}) - q - 2$. □

Theorem 3 *If $\overline{B}_+$ is a q-complete space, then*

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_+, \mathcal{F})$$

for $q \leq r \leq p(\mathcal{F}) - q - 2$ and the homomorphism

$$H^{q-1}(\overline{B}_+, \mathcal{F}) \oplus H^{q-1}(\overline{C}, \mathcal{F}) \longrightarrow H^{q-1}(\overline{C}_+, \mathcal{F}) \tag{4}$$

is surjective for $p(\mathcal{F}) \geq 2q + 1$.

If $\overline{B}_+$ is a 1-complete space and $p(\mathcal{F}) \geq 3$, the homomorphism

$$H^0(\overline{B}_+, \mathcal{F}) \longrightarrow H^0(\overline{C}_+, \mathcal{F})$$

is surjective.

Proof Since by hypothesis $\overline{B}_+$ is a q-complete space, $H^r(\overline{B}_+, \mathcal{F}) = \{0\}$ for $q \leq r$ [1, Théorème 5]. From (3) it follows that $H^r(\overline{B}_+ \cup \overline{C}_+, \mathcal{F}) = \{0\}$ for $q \leq r \leq p(\mathcal{F}) - q - 1$. Thus, the Mayer-Vietoris sequence (1) implies that $H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_+, \mathcal{F})$ for $q \leq r \leq p(\mathcal{F}) - q - 2$ and that the homomorphism (4) is surjective if $p(\mathcal{F}) \geq 2q + 1$.

In particular, if $q = 1$ and $p(\mathcal{F}) \geq 3$ the homomorphism

$$H^0(\overline{B}_+, \mathcal{F}) \oplus H^0(\overline{C}, \mathcal{F}) \longrightarrow H^0(\overline{C}_+, \mathcal{F})$$

is surjective i.e. every section $\sigma \in H^0(\overline{C}_+, \mathcal{F})$ is a difference $\sigma_1 - \sigma_2$ of two sections $\sigma_1 \in H^0(\overline{B}_+, \mathcal{F}), \sigma_2 \in H^0(\overline{C}, \mathcal{F})$. Since B_ε is Stein, the cohomology group with compact supports $H^1_k(B_\varepsilon, \mathcal{F})$ is zero, and so the Mayer–Vietoris compact supports cohomology sequence implies that the restriction homomorphism

$$H^0(\overline{B}_c, \mathcal{F}) \longrightarrow H^0(\overline{B}_c \setminus B_\varepsilon, \mathcal{F}) = H^0(\overline{C}, \mathcal{F})$$

is surjective, hence $\sigma_2 \in H^0(\overline{C}, \mathcal{F})$ is restriction of $\widetilde{\sigma}_2 \in H^0(B_c, \mathcal{F})$. So σ is restriction to $\overline{C}_+$ of $(\sigma_1 - \widetilde{\sigma}_{2|\overline{B}_+}) \in H^0(\overline{B}_+, \mathcal{F})$, and the restriction homomorphism is surjective. □

Corollary 4 *Let $\overline{B}_+$ be a 1-complete space and $p(B_c) \geq 3$. Then every holomorphic function on $\overline{C}_+$ extends holomorphically on $\overline{B}_+$.*

2.2 Open q-coronae

Most of the Theorems and Corollaries of the previous section still hold in the open case and their proofs are very similar. First we give the proof of the extension results using directly Theorem 3. We have to assume that H is the zero set of a pluriharmonic function h and we define B_c, C, B_+, B_-, C_+ and C_- as we did before.

Let us suppose B_+ is 1-complete and $p(\mathcal{F}) \geq 3$. Let $s \in H^0(C_+, \mathcal{F})$. For all $\epsilon > 0$, we consider the closed semi 1-corona

$$\overline{C}_\epsilon = \overline{B_{\varepsilon+\epsilon,c} \cap \{h > \epsilon\}} \subset C_+$$

Let $\sigma_\epsilon = s_{|\overline{C}_\epsilon}$. By Theorem 3 (applied to $\overline{C}_\epsilon$, $H_\epsilon = \{h = \epsilon\}$), we obtain that σ_ϵ extends to a section $\widetilde{\sigma}_\epsilon \in H^0(\overline{B}_\epsilon, \mathcal{F})$, where $\overline{B}_\epsilon = \overline{B_+ \cap \{h > \epsilon\}}$. Since $B_+ = \cup_\epsilon \overline{B}_\epsilon$, if for all $\epsilon_2 > \epsilon_1 > 0$,

$$\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} = \widetilde{\sigma}_{\epsilon_2} \tag{5}$$

the sections $\widetilde{\sigma}_\epsilon$ can be glued toghether to a section $\sigma \in H^0(B_+, \mathcal{F})$ extending s.

Let $\epsilon_1, \epsilon_2, \epsilon_2 > \epsilon_1 > 0$, be fixed. We have to show that (5) holds. By definition,

$$\left(\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}\right)_{|\overline{C}_{\epsilon_2}} = s - s = 0.$$

Thus, the support of $\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}$, S, is an analytic set contained in $\overline{B}_{\epsilon_2} \setminus C_{\epsilon_2}$. Let us consider the family $(\phi_\lambda = \lambda(\varphi - \epsilon_2) + (1 - \lambda)(h - \epsilon_2))_{\lambda \in [0,1]}$ of strictly plurisubharmonic functions. Let $\overline{\lambda}$ be the smallest value of λ for which $\{\phi_\lambda = 0\} \cap S \neq \varnothing$. Then $\{\phi_{\overline{\lambda}} < 0\} \cap B_+ \subset B_+$ is a Stein domain in which the analytic set S intersects the boundary; so the maximum principle for plurisubharmonic functions and the strict plurisubharmonicity of $\phi_{\overline{\lambda}}$ together imply that $\{\phi_{\overline{\lambda}} = 0\} \cap S$ is a set of isolated points in S. By repeating the argument, we show that S has no components of positive dimension. Hence $\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}$ is zero outside a set of isolated points. Since $p(\mathcal{F}) \geq 3$, the only section of $\mathcal{F}$ with compact support is the zero-section [3, Théorème 3.6 (a), p. 46], and so $\widetilde{\sigma}_{\epsilon_1|_{\overline{B}_{\epsilon_2}}} - \widetilde{\sigma}_{\epsilon_2}$ is zero.

Hence, there exists a section $\sigma \in H^0(B_+, \mathcal{F})$ such that $\sigma_{|C_+} = s$. Thus we have proved the following

Theorem 5 *If a B_+ is 1-complete space, $\mathcal{F}$ a coherent sheaf on B_+ with $p(\mathcal{F}) \geq 3$, the homomorphism*

$$H^0(B_+, \mathcal{F}) \longrightarrow H^0(C_+, \mathcal{F})$$

is surjective.

In particular,

Corollary 6 *If B_+ is a 1-complete space and $p(B_c) \geq 3$, every holomorphic function on C_+ can be holomorphically extended on B_+.*

Theorem 7 *Let* $\mathrm{Sing}(B_c) = \varnothing$. *Let $\mathcal{F} \in$* $\mathrm{Coh}(B_c)$. *Then the image of the homomorphism*

$$H^r(B_+, \mathcal{F}) \oplus H^r(C, \mathcal{F}) \longrightarrow H^r(C_+, \mathcal{F})$$

defined by $(\xi, \eta) \mapsto \xi_{|C_+} - \eta_{|C_+}$ *has finite codimension for* $q-1 \le r \le p(\mathcal{F}) - q - 2$. *For* $q = 1$ *the thesis holds true also dropping the assumption* $\mathrm{Sing}(B_c) = \varnothing$.

Proof Consider the Mayer-Vietoris sequence applied to the open sets B_+ and C

$$\begin{aligned} \cdots \to H^r(B_+ \cup C, \mathcal{F}) \to H^r(B_+, \mathcal{F}) \oplus H^r(C, \mathcal{F}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^r(C_+, \mathcal{F}) \to H^{r+1}(B_+ \cup C, \mathcal{F}) \to \cdots, \end{aligned} \tag{6}$$

$\delta(a \oplus b) = a_{|C_+} - b_{|C_+}$. $B_+ \cup C = B_c \smallsetminus K_0$ where $K_0 = \overline{B}_- \cap \overline{B}_\varepsilon$. K_0 has a q-complete neighbourhoods system and so the local cohomology groups $H^r_{K_0}(B_c, \mathcal{F})$ are zero for $q \le r \le p(\mathcal{F}) - q$ [4] (in the general case, for $q = 1$, see [3, Lemme 2.3, p. 29]).

Then, from the local cohomology exact sequence

$$\begin{aligned} \cdots \to H^r_{K_0}(B_c, \mathcal{F}) \to H^r(B_c, \mathcal{F}) \to \\ \to H^r(B_c \smallsetminus K_0, \mathcal{F}) \to H^{r+1}_{K_0}(B_c, \mathcal{F}) \to \cdots \end{aligned} \tag{7}$$

it follows that

$$H^r(B_c, \mathcal{F}) \xrightarrow{\sim} H^r(B_c \smallsetminus K_0, \mathcal{F}), \tag{8}$$

for $q \le r \le p(\mathcal{F}) - q - 1$.

Since B_c is q-pseudoconvex,

$$\dim_{\mathbb{C}} H^r(C, \mathcal{F}) < \infty$$

for $q \le r$ [1, Théorème 11], and so

$$\dim_{\mathbb{C}} H^r(B_c \smallsetminus K_0, \mathcal{F}) < \infty$$

for $q \le r \le p(\mathcal{F}) - q - 1$.

From (6) we see that $\dim_{\mathbb{C}} H^r(B_c \smallsetminus K_0, \mathcal{F}) = \dim_{\mathbb{C}} H^r(B_+ \cup C, \mathcal{F})$ is greater than or equal to the codimension of the homomorphism δ. □

Corollary 8 *Under the same assumption of Theorem 7, if* $K \cap H = \varnothing$,

$$\dim_{\mathbb{C}} H^r(C_+, \mathcal{F}) < \infty$$

for $q \le r \le p(\mathcal{F}) - q - 2$.

Proof The proof is similar to that of Corollary 2. □

Theorem 9 *Suppose that* $\mathrm{Sing}(B_c) = \varnothing$ *and* B_+ *is a* q*-complete space, then*

$$H^r(C, \mathcal{F}) \xrightarrow{\sim} H^r(C_+, \mathcal{F})$$

for $q \le r \le p(\mathcal{F}) - q - 2$ *and the homomorphism*

$$H^{q-1}(B_+, \mathcal{F}) \oplus H^{q-1}(C, \mathcal{F}) \longrightarrow H^{q-1}(C_+, \mathcal{F}) \tag{9}$$

is surjective if $p(\mathcal{F}) \ge 2q + 1$. *If* $q = 1$, *both results hold true for an arbitrary complex space* B_c.

Proof The proof is similar to that of Theorem 3. □

2.3 Corollaries of the extension theorems

From now on, unless otherwise stated, by B, B_+, B_ε, C and C_+ we denote both the open sets and their closures, and we suppose that $H = \{h = 0\}$, h pluriharmonic.

2.3.1 Let $f \in H^0(C_+, \mathcal{O}^*)$. Under the hypothesis of Corollaries 4 and 6, both f and $1/f$ extend holomorphically on B_+. Hence:

Corollary 10 *If B_+ is a 1-complete space and $p(B_c) \geq 3$, the restriction homomorphism*

$$H^0(B_+, \mathcal{O}^*) \longrightarrow H^0(C_+, \mathcal{O}^*)$$

is surjective.

2.3.2 In Theorems 3 and 5 we have estabilished the isomorphism

$$H^r(C, \mathcal{F}) \xrightarrow{\sim} H^r(C_+, \mathcal{F}).$$

In some special cases this leads to vanishing-cohomology theorems for C_+. An example is provided by a q-corona C which is contained in an affine variety. In such a situation, we have that $H^r(C, \mathcal{F}) = \{0\}$, for $q \leq r \leq p(\mathcal{F}) - q - 2$ [2], and consequently $H^r(C_+, \mathcal{F}) = \{0\}$ in the same range of r.

2.3.3 Let X be a Stein space. Let $H = \{h = 0\} \subset X$ be the zero set of a pluriharmonic function, and let S be a real hypersurface of X with boundary, such that $S \cap H = bS = bA$, where A is an open set in H. Let $D \subset X$ be the relatively compact domain bounded by $S \cup A$. In [13] it is proved that, for $X = \mathbb{C}^n$, CR-functions on S extend holomorphically to D. As a corollary of the previous theorems, we can obtain a similar result for section of a coherent sheaf on an arbitrary Stein space X.

Let us consider the connected component Y of $X \smallsetminus H$ containing D, the closure $\overline{D}$ of D in Y, and let $F = Y \smallsetminus D$ and $S_Y = S \cap Y$. For every coherent sheaf $\mathcal{F}$ on X, with $p(\mathcal{F}) \geq 3$ we have the Mayer–Vietoris exact sequence

$$\cdots \rightarrow H^0(\overline{D}, \mathcal{F}) \oplus H^0(F, \mathcal{F}) \rightarrow H^0(S_Y, \mathcal{F}) \rightarrow H^1(Y, \mathcal{F}) \rightarrow \cdots$$

Since Y is Stein, $H^1(Y, \mathcal{F})$ is zero and every section σ on S_Y is a difference $s_1 - s_2$, where $s_1 \in H^0(\overline{D}, \mathcal{F})$ and $s_2 \in H^0(F, \mathcal{F})$. By choosing an ε big enough so that S is contained in the ball $B_\varepsilon(x_0)$ of radius ε of X centred in x_0, we can apply Theorem 5 to the semi 1-corona $C_+ = Y \smallsetminus (B_\varepsilon \cap Y)$, to extend $s_{2|C_+}$ to a section $\tilde{s}_2$ defined on Y. In order to conclude that $s_1 - \tilde{s}_{2|\overline{D}}$ extends the section σ, we have to prove that $s_{2|F} - \tilde{s}_{2|F} = 0$.

As before, we consider the set $\Sigma = \left\{s_{2|F} - \tilde{s}_{2|F} \neq 0\right\} \subset B_\varepsilon \cap Y$ and conclude that Σ is a set of isolated points. Since $p(\mathcal{F}) \geq 3$, $\mathcal{F}$ has no non-zero section with compact support [3, Théorème 3.6 (a), p. 46]. Thus $\Sigma = \varnothing$ and we have obtained the following:

Corollary 11 *Let X be a Stein space. Let $H = \{h = 0\} \subset X$ be the zero set of a pluriharmonic function, and S a real hypersurface of X with boundary, such that $S \cap H = bS = bA$, where A is an open set in H. Let $D \subset X$ be the relatively compact domain bounded by $S \cup A$ and $\mathcal{F}$ a coherent sheaf with $p(\mathcal{F}) \geq 3$. All sections of $\mathcal{F}$ on S extend (uniquely) to D.*

We can go further:

Corollary 12 *Let X be a Stein manifold, $\mathcal{F}$ a coherent sheaf on X such that $p(\mathcal{F}) \geq 3$, D a bounded domain and K a compact subset of bD such that $bD \smallsetminus K$ is smooth. Assume that K is $\mathcal{O}(D)$-convex,* i.e.

$$K = \left\{ z \in \overline{D} \; : \; |f(z)| \leq \max_K |f| \right\}.$$

Then every section of $\mathcal{F}$ on $bD \smallsetminus K$ extends to D.

Proof We recall that since U is an open subset of a Stein manifold there exists an envelope of holomorphy $\widetilde{U}$ of U (cfr. [6]); $\widetilde{U}$ is a Stein domain $\pi_U : \widetilde{U} \to X$ over X and there exists and open embedding $j : U \to \widetilde{U}$ such that $\pi_U \circ j = id_U$ and $J^* : \mathcal{O}(\widetilde{U}) \to \mathcal{O}(U)$ is an isomorphism. In particular, $\pi_U^* \mathcal{F}$ is a coherent sheaf with the same depth as $\mathcal{F}$ which extends $\mathcal{F}_{|U}$.

Let us fix an arbitrary point $x \in D$. We need to show that any given section $\sigma \in H^0(bD \smallsetminus K, \mathcal{F})$ extends to a neighbourhood of x. Since $x \notin K = \widehat{K}$, there exists an holomorphic function f, defined on a neighbourhood U of $\overline{D}$, such that $|f(x)| > \max_K |f(z)|$.

Then σ extends to a section $\widetilde{\sigma} \in H^0(\pi^{-1}(D \smallsetminus K), \mathcal{F})$. Let $\widetilde{f}$ be the holomorphic extension of f to $\widetilde{U}$. The hypersurface

$$H = \left\{ z \in \widetilde{U} : |\widetilde{f}(z)| = \max_K |\widetilde{f}| \right\}$$

is the zero-set of a pluriharmonic function and, by construction,

$$x \in \widetilde{D}_+ = \left\{ z \in \widetilde{U} : |\widetilde{f}(z)| > \max_K |\widetilde{f}| \right\}.$$

Now we are in the situation of Corollary 11 so $\widetilde{\sigma}$ extends to a section on $\widetilde{D}_+$. Since $x \in \widetilde{D}_+$, this ends the proof. □

3 Extension of divisors and analytic sets of codimension one

First of all, we give an example in dimension $n = 2$ of a regular complex curve of C_+ which does not extend on B_+. Hence, not every divisor on C_+ extends to a divisor on B_+.

Example Using the same notation as before, let B_c be the ball $\{|z_1|^2 + |z_2|^2 < c\}$, $c > 2$, in $\mathbb{C}^2$, and H be the hyperplane $\{x_2 = 0\}$ $(z_j = x_j + iy_j)$. Let $2 < \varepsilon < c$, $C = B_c \smallsetminus \overline{B}_\varepsilon$, $B_+ = B_c \cap \{x_2 > 0\}$, $C_+ = C \cap \{x_2 > 0\}$.

Consider the connected irreducible analytic set of codimension one

$$A = \{(z_1, z_2) \in B_+ \ : \ z_1 z_2 = 1\}$$

and its restriction A_C to C_+. If A_C has two connected components, A_1 and A_2 and we try to extend A_1 (analytic set of codimension one on C_+) to B_+, its restriction to C_+ will contain also A_2. So A_1 is an analytic set of codimension one on C_+ that does not extend on B_+.

Thus let us prove that A_C has indeed two connected components. A point of A (of A_C) can be written as $z_1 = \rho e^{i\theta}$, $z_2 = \frac{1}{\rho} e^{-i\theta}$, with $\rho \in \mathbb{R}^+$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, points in A_C satisfy

$$2 < \varepsilon < \rho^2 + \frac{1}{\rho^2} < c \ \Rightarrow \ 2 < \sqrt{\varepsilon + 2} < \rho + \frac{1}{\rho} < \sqrt{c + 2}.$$

Since $f(\rho) = \rho + 1/\rho$ is monotone decreasing up to $\rho = 1$ (where $f(1) = 2$), and then monotone increasing, there exist a and b such that the inequalities are satisfied when

$a < \rho < b < 1$, or when $1 < 1/b < \rho < 1/a$. A_C is thus the union of the two disjoint open sets

$$A_1 = \left\{ \left(\rho e^{i\theta}, \tfrac{1}{\rho} e^{-i\theta} \right) \in \mathbb{C}^2 \;\middle|\; a < \rho < b,\ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \right\};$$

$$A_2 = \left\{ \left(\rho e^{i\theta}, \tfrac{1}{\rho} e^{-i\theta} \right) \in \mathbb{C}^2 \;\middle|\; a < \tfrac{1}{\rho} < b,\ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \right\}.$$

The aim of this section is to prove an extension theorem for divisors, i.e. to prove that, under certain hypothesis, the homomorphism

$$H^0(B_+, \mathcal{D}) \to H^0(C_+, \mathcal{D}) \tag{10}$$

is surjective.

In order to get this result, we observe that from the exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0 \tag{11}$$

we get the commutative diagram (horizontal lines are exact)

$$\begin{tikzcd}
H^0(B_+, \mathcal{M}^*) \arrow[r] \arrow[d, "\alpha"'] & H^0(B_+, \mathcal{D}) \arrow[r] \arrow[d, "\beta"'] & H^1(B_+, \mathcal{O}^*) \arrow[r] \arrow[d, "\gamma"'] & H^1(B_+, \mathcal{M}^*) \arrow[d, "\delta"'] \\
H^0(C_+, \mathcal{M}^*) \arrow[r] & H^0(C_+, \mathcal{D}) \arrow[r] & H^1(C_+, \mathcal{O}^*) \arrow[r] & H^1(C_+, \mathcal{M}^*)
\end{tikzcd}$$

Thus, in view of the "five lemma", in order to conclude that β is surjective it is sufficient to show that α and γ are surjective, and δ is injective.

Lemma 13 *If* $\mathrm{Sing}(B_+) = \varnothing$, B_c *is* 1*-complete and* $p(B_c) \geq 3$, *then* α *is surjective.*

Proof Let f be a meromorphic invertible function on C_+. Since C_+ is an open set of the Stein manifold B_+, $f = f_1 f_2^{-1}$, $f_1, f_2 \in H^0(C_+, \mathcal{O})$. By Corollary 4 (6), f_1 and f_2 extend to holomorphic functions on B_+ and consequently f extends on B_+ as well. □

Lemma 14 *Assume that the restriction* $H^2(B_+, \mathbb{Z}) \to H^2(C_+, \mathbb{Z})$ *is surjective. If* B_c *is* 1*-complete and* $p(B_c) \geq 4$, *then* γ *is surjective.*

We remark that if $H^2(C_+, \mathbb{Z}) = \{0\}$ the first condition is satisfied.

Proof From the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{12}$$

we get the commutative diagram (horizontal lines are exact)

$$\begin{tikzcd}
H^1(B_+, \mathcal{O}) \arrow[r] \arrow[d, "f_2"'] & H^1(B_+, \mathcal{O}^*) \arrow[r] \arrow[d, "\gamma"'] & H^2(B_+, \mathbb{Z}) \arrow[r] \arrow[d, "f_4"'] & H^2(B_+, \mathcal{O}) \arrow[d, "f_5"'] \\
H^1(C_+, \mathcal{O}) \arrow[r] & H^1(C_+, \mathcal{O}^*) \arrow[r] & H^2(C_+, \mathbb{Z}) \arrow[r] & H^2(C_+, \mathcal{O})
\end{tikzcd}$$

where $H^1(B_+, \mathcal{O}) = H^2(B_+, \mathcal{O}) = \{0\}$ because B_+ is Stein, and f_4 is surjective by hypothesis. Thus in order to prove that γ is surjective by the "five lemma" it is sufficient to show that f_2 is surjective, i.e. that $H^1(C_+, \mathcal{O}) = \{0\}$.

Since $p(B_c) \geq 4$, by Theorem 3 (9) it follows that

$$H^1(C, \mathcal{O}) \xrightarrow{\sim} H^1(C_+, \mathcal{O}). \tag{13}$$

Consider the local, respectively compact supports, cohomology exact sequence

$$H^1_{\overline{B}_\varepsilon}(B_c, \mathcal{O}) \longrightarrow H^1(B_c, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O}) \longrightarrow H^2_{\overline{B}_\varepsilon}(B_c, \mathcal{O})$$

$$H^1_k(B_\varepsilon, \mathcal{O}) \longrightarrow H^1(B_c, \mathcal{O}) \longrightarrow H^1(C, \mathcal{O}) \longrightarrow H^2_k(B_\varepsilon, \mathcal{O})$$

Since B_c is Stein, $H^1(B_c, \mathcal{O}) = \{0\}$ and $H^r_k(B_\varepsilon, \mathcal{O}) = H^r_{\overline{B}_\varepsilon}(B_c, \mathcal{O}) = \{0\}$ for $1 \leq r \leq p(B_\varepsilon) - 1$ [4]. In particular, since $p(B_\varepsilon) \geq p(B_c) \geq 4$, it follows that

$$\{0\} = H^1(B_c, \mathcal{O}) \xrightarrow{\sim} H^1(C, \mathcal{O}). \tag{14}$$

(13) and (14) give

$$\{0\} = H^1(B_c, \mathcal{O}) \xrightarrow{\sim} H^1(C, \mathcal{O}) \xrightarrow{\sim} H^1(C_+, \mathcal{O}).$$

and this proves the lemma. □

In the case $H^2(C_+, \mathbb{Z}) = \{0\}$ we remark that from the proof of Lemma 14 it follows that the sequence

$$\{0\} \longrightarrow H^1(C_+, \mathcal{O}^*) \longrightarrow \{0\}$$

is exact, that is $H^1(C_+, \mathcal{O}^*) = \{0\}$. Hence, the commutative diagram relative to (11) becomes (horizontal lines are exact)

$$\begin{tikzcd}
H^0(B_+, \mathcal{M}^*) \arrow[r] \arrow[d, "\alpha"'] & H^0(B_+, \mathcal{D}) \arrow[r] \arrow[d, "\beta"'] & H^1(B_+, \mathcal{O}^*) \arrow[d, "\gamma"'] \\
H^0(C_+, \mathcal{M}^*) \arrow[r] & H^0(C_+, \mathcal{D}) \arrow[r] & \{0\}
\end{tikzcd} \tag{15}$$

and it is then easy to see that a divisor on C_+ can be extended to a divisor on B_+.

Thus we have proved the following:

Theorem 15 *Let B_c be 1-complete, $p(B_c) \geq 4$, and C_+ satisfy the topological condition $H^2(C_+, \mathbb{Z}) = \{0\}$. Then, if $\operatorname{Sing}(B_+) = \varnothing$, all divisors on C_+ extend (uniquely) to divisors on B_+.*

Corollary 16 *Let B_c be 1-complete, $p(B_c) \geq 4$, $\operatorname{Sing}(B_+) = \varnothing$, and ξ be a divisor on C_+ with zero Chern class in $H^2(C_+, \mathbb{Z})$. Then ξ extends (uniquely) to a divisor on B_+.*

Proof Use diagram (15). □

Theorem 17 *Assume that $H^2(C_+, \mathbb{Q}) = \{0\}$. If $\operatorname{Sing}(B_+) = \varnothing$, B_c is 1-complete and $p(B_c) \geq 4$, then all analytic sets of codimension 1 on C_+ extend to analytic sets on B_+.*

Proof Let A be an analytic set of codimension 1 in C_+. Since B_+ is a Stein manifold, C_+ is locally factorial, and so there exists a divisor ξ on C_+ with support A. Since $H^2(C_+,\mathbb{Q}) = \{0\}$, there exists $n \in \mathbb{N}$ such that $nc_2(\xi) = 0 \in H^2(C_+,\mathbb{Z})$. Hence $n\xi$ has zero Chern class in $H^2(C_+,\mathbb{Z})$, and so, by Corollary 16, $n\xi$ can be extended to a divisor $\widetilde{n\xi}$ on B_+. The support of $\widetilde{n\xi}$ is an analytic set $\widetilde{A}$ which extends to B_+ the support A of $n\xi$. □

In Theorem 15 the condition $H^2(C_+,\mathbb{Z}) = \{0\}$ can be relaxed and replaced by the weaker one: the restriction map $H^2(B_+,\mathbb{Z}) \to H^2(C_+,\mathbb{Z})$ is surjective. We need the following

Lemma 18 *δ is injective.*

Proof First we prove the lemma for C_+ closed. Let $\xi \in H^1(\overline{B}_+,\mathcal{M}^*)$ be such that $\xi_{|\overline{C}_+} = 0$. Consider the set

$$A = \{\eta \in [0,\varepsilon] \,:\, \xi_{|\overline{B}_+ \smallsetminus \overline{B}_\eta} = 0\}.$$

If we prove that $0 \in A$, we are done, because $0 = \xi_{|\overline{B}_+ \smallsetminus \overline{B}_0} = \xi_{|\overline{B}_+} = \xi$. Obviously $\eta_0 \in A$ implies $\forall \eta > \eta_0$, $\eta \in A$.

$A \neq \varnothing$. Since $C_+ = B_+ \smallsetminus \overline{B}_\xi$ and $\xi_{|\overline{C}_+} = 0$, $\varepsilon \in A$.

A is closed. If $\eta_n \in A$, for all n, and $\eta_n \searrow \eta_\infty$, $\overline{B}_+ \smallsetminus \overline{B}_{\eta_\infty} = \cup_n(\overline{B}_+ \smallsetminus \overline{B}_{\eta_n})$, hence $\xi_{|\overline{B}_+ \smallsetminus \overline{B}_{\eta_n}} = 0$ for all n implies $\xi_{|\overline{B}_+ \smallsetminus \overline{B}_{\eta_\infty}} = 0$, i.e. $\eta_\infty \in A$.

A is open. Suppose $0 < \eta_0 \in A$. We denote $C_{\eta_0} = \overline{B}_+ \smallsetminus \overline{B}_{\eta_0}$. Let $\mathcal{A}$ be the family of open covering $\{U_i\}_{i \in I}$ of $\overline{B}_+$ such that:

(α) U_i is isomorphically equivalent to a holomorphy domain in $\mathbb{C}^n$;
(β) if $U_i \cap bB_{\eta_0} \neq \varnothing$, the restriction homomorphism

$$H^0(U_i,\mathcal{O}) \to H^0(U_i \cap C_{\eta_0},\mathcal{O})$$

is bijective;
(γ) $U_i \cap U_j$ is simply connected.

$\mathcal{A}$ is not empty and it is cofinal in the set of open coverings of $\overline{B}_+$ [1, Lemma 2, p. 222]. Let $\mathcal{U} = \{U_i\}_{i \in I} \in \mathcal{A}$, and $\{f_{ij}\} \in Z^1(\mathcal{U},\mathcal{M}^*)$ be a representative of ξ. Let $W_i = U_i \cap C_{\eta_0}$. Since $\eta_0 \in A$, if $W_i \cap W_j \neq \varnothing$, $f_{ij|W_i \cap W_j} = f_i f_j^{-1}$ ($f_\nu \in H^0(W_\nu,\mathcal{M}^*)$). By ($\alpha$), $f_\nu = p_\nu q_\nu^{-1}$, $p_\nu, q_\nu \in H^0(W_\nu,\mathcal{O})$. By ($\beta$), both p_ν and q_ν can be holomorphically extended on U_ν, with $\widetilde{p}_\nu$ and $\widetilde{q}_\nu$. Hence we have $f_{ij} = \widetilde{p}_i\widetilde{q}_i^{-1}(\widetilde{p}_j\widetilde{q}_j^{-1})^{-1}$ on $U_i \cap U_j$ (which is simply connected, so that there is no polidromy). So $\xi = 0$ in an open neighbourhood U of C_{η_0} and, by compactness, there exists $\epsilon' > 0$ such that $C_{\eta_0 - \epsilon'} \subset U$. So $\eta_0 - \epsilon' \in A$ and consequently A is open.

Thus $A = [0,\varepsilon]$, and the lemma is proved if C_+ is closed.

If C_+ is open, we consider C_+ as a union of the closed semi 1-coronae

$$\overline{C}_\epsilon = \overline{B_{\varepsilon+\epsilon',c} \cap \{h > \epsilon'\}} \subset C_+.$$

Let $\xi \in H^1(B_+,\mathcal{M}^*)$ be such that $\xi_{|C_+} = 0$. Then $\xi_{|\overline{C}'_\epsilon} = 0$, for all $\epsilon' > 0$. Consequently, from what we have already proved, $\xi_{|\overline{B}'_\epsilon} = 0$, where $\overline{B}_\epsilon = \overline{B_+ \cap \{h > \epsilon'\}}$. Since $\cup'_\epsilon \overline{B}'_\epsilon = B_+$, $\xi = 0$ and the lemma is proved. □

Lemmas 13, 14 and 18 lead to the following generalization of Theorem 15:

Theorem 19 *Assume that the restriction* $H^2(B_+, \mathbb{Z}) \to H^2(C_+, \mathbb{Z})$ *is surjective. If* $\mathrm{Sing}(B_+) = \varnothing$, B_c *is* 1*-complete and* $p(B_c) \geq 4$, *then all divisors on* C_+ *extend to divisors on* B_+.

References

1. Andreotti, A., Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. Fr. **90**, 193–259 (1962)
2. Andreotti, A., Tomassini, G.: A remark on the vanishing of certain cohomology groups. Comp. Math. **21**, 417–430 (1969)
3. Bănică, C., Stănăşilă, O.: Méthodes algébriques dans la théorie globale des espaces complexes. Editura Academiei et Gauthier-Villars, Bordas Paris (1977)
4. Cartan, H.: Faisceaux analytiques cohérents. In: corso C.I.M.E. Funzioni e varietà complesse, Varenna (1963)
5. Della Sala, G., Saracco, A.: Non compact boundaries of complex analytic varieties. Int. J. Math., arXiv math.CV/0503430 (to appear)
6. Docquier, F., Grauert, H.: Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten. Math. Ann. **140**, 94–123 (1960)
7. Frisch, J., Guenot, J.: Prolongement de faisceaux analytiques cohérents. Invent. Math. **7**, 321–343 (1969)
8. Harvey, F.R., Lawson, H.B.: On boundaries of complex analytic varieties I. Ann. Math. **102**, 223–290 (1975)
9. Jöricke, B.: Boundaries of singularity sets, removable singularities, and CR-invariant subsets of CR-manifolds. J. Geom. Anal. **9**, 257–300 (1999)
10. Laurent-Thiébaut, C.: Sur l'extension des fonctions CR dans une variété de Stein. Ann. Math. Pura Appl. **150**, 141–151 (1988)
11. Laurent-Thiébaut, C., Porten, E.: Analytic extension from non-pseudoconvex boundaries and $A(D)$-convexity. Ann. Inst. Fourier (Grenoble) **53**, 847–857 (2003)
12. Lupacciolu, G.: A theorem on holomorphic extension of CR-functions. Pac. J. Math. **124**, 177–191 (1986)
13. Lupacciolu, G., Tomassini, G.: Un teorema di estensione per le CR-funzioni. Ann. Math. Pura Appl. **137**, 257–263 (1984)
14. Perotti, A.: Extension of CR-forms and related problems. Rend. Sem. Mat. Univ. Padova **77**, 37–55 (1987)
15. Serre, J.P.: Prolongement de faisceaux analytiques cohérents. Ann. Inst. Fourier (Grenoble) **16**, 363–374 (1966)
16. Siu, Y.T.: Extension problem in several complex variables. In: Proceedings of the International Congress IMU, Helsinki, pp. 669–673 (1978)
17. Siu, Y.T., Trautmann, G.: Gap-sheaves and extension of coherent analytic subsheaves. Lecture Notes in Mathematics, vol. 172. Springer, Berlin (1971)
18. Stout, E.L.: Removable singularities for the boundary values of holomorphic functions of several complex variables. In: Proceedings of the Mittag-Leffler special year in complex variables, Stockholm 1987–1988. Princeton University Press, Princeton, pp. 600–629 (1993)
19. Tomassini, G.: Inviluppo d'olomorfia e spazi pseudoconcavi. Ann. Mat. Pura Appl. **87**, 59–86 (1970)
20. Tomassini, G.: Extension d'objèts CR. Math. Z. **194**, 471–486 (1987)