Abstract

We transfer several characterizations of hyperbolic convex domains given in a recent joint paper by Bracci and one of the authors to analogous one for $\mathbb{C}$-convex domains.

Key words: $\mathbb{C}$-convexity, hyperbolicity, completeness, barrier

2000 Mathematics Subject Classification: 32Q45

Despite the fact that convexity is not an invariant property in complex analysis, bounded convex domains in $\mathbb{C}^N$ have been intensively studied as prototypes for the general situation.

Natural generalizations of the notion of convexity is $\mathbb{C}$-convexity (cf. [1,5]) which, although non-invariant, reflects the underlying complex vector space structure. A domain $D \subset \mathbb{C}^N$ is called $\mathbb{C}$-convex if any non-empty intersection of $D$ with a complex line is contractible.

We point out the following

Proposition 1 [7]. Let $D \subset \mathbb{C}^N$ be a $\mathbb{C}$-convex domain. Then there exist a unique $k$ ($0 \leq k \leq N$) and a unique $\mathbb{C}$-convex $D' \subset \mathbb{C}^k$ containing no complex lines, such that up to a linear change of coordinates, $D = D' \times \mathbb{C}^{N-k}$. Moreover, $D'$ is biholomorphic to a bounded domain and it is $\epsilon$-finitely compact (i.e. the balls with respect to the Carathéodory distance of $D'$ are relatively compact in $D'$).

Recall that the Carathéodory pseudodistance is the smallest pseudodistance decreasing under holomorphic mapping and coinciding with the Poincaré distance on the unit disc in $\mathbb{C}$, while the Kobayashi pseudodistance is the largest one with these properties.

Based on Proposition 1, in this note we generalize several characterizations of hyperbolicity obtained in [2] for convex domains to $\mathbb{C}$-convex domains (for the definitions we refer the reader to e.g. [6]).
Theorem 1. Let \( D \subset \mathbb{C}^N \) be a (possibly unbounded) \( \mathbb{C} \)-convex domain. Then the following conditions are equivalent:

1. \( D \) is biholomorphic to a bounded domain;
2. \( D \) is complete with respect to the Carathéodory distance;
3. \( D \) is complete with respect to the Kobayashi distance;
4. \( D \) is Kobayashi hyperbolic;
5. \( D \) admits complete Bergman metric;
6. \( D \) is taut (i.e. the family \( \mathcal{O}(D, D) \) is normal, where \( D \subset \mathbb{C} \) is the unit disc);
7. \( D \) is hyperconvex (i.e. \( D \) has a negative plurisubharmonic (psh) exhaustion function);
8. \( D \) is Brody hyperbolic (i.e. \( D \) contains no nonconstant entire curves);
9. \( D \) contains no complex lines;
10. \( D \) has \( N \) linearly independent separating complex lines (i.e. lines passing through boundary points of \( D \) and disjoint from \( D \));
11. \( D \) has a strong psh barrier at \( \infty \) (i.e. a psh function \( \varphi \) such that \( \limsup_{z \to a} \varphi(z) < 0 = \lim_{z \to 1} \varphi(z) \) for any finite \( a \in D \));
12. \( D \) has an antipeak function at infinity (in sense of Gaussier [4], i.e. a psh function \( \varphi < 0 \) such that \( \liminf_{z \to a} \varphi(z) > -\infty = \lim_{z \to 1} \varphi(z) \) for any finite \( a \in \overline{D} \)).

Proof. The implication (i) \( \Rightarrow \) (1), 2 \( \leq \) i \( \leq \) 12, trivially follows by Proposition 1 (if (1) does not hold, then \( D \) contains a complex line);
1. (1) \( \Rightarrow \) (2) also follows by Proposition 1;
2. (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (8) \( \Rightarrow \) (9) trivially hold for any domain;
3. (2) \( \Rightarrow \) (7) is true for any domain, since \( \tanh c_D - 1 \) is a negative psh exhaustion function of \( D \) (here \( c_D \) stands for the Carathéodory distance of \( D \));
4. (7) \( \Rightarrow \) (6) and (7) \( \Rightarrow \) (5) hold also for any domain; see [6] and [3], respectively;
5. To show that (1) \( \Rightarrow \) (10), (11), (12), we shall use that up to a linear change of coordinates, \( D \subset \prod_{j=1}^N D_j \), where any \( D_j \) is biholomorphic to \( \mathbb{D} \) (see [7]). This immediately gives that (1) \( \Rightarrow \) (10).
6. Assume now that \( D_1, \ldots, D_N \) are unbounded. Since \( D_j \) is hyperconvex, it admits a strong subharmonic barrier at any boundary point (including \( \infty \)) by Bouligand’s lemma for unbounded planar domains (cf. [8]; see also Remark (b)). If \( \varphi_j \) denotes the barrier at \( \infty \), then \( \varphi = \max_{1 \leq j \leq N} \varphi_j \) is a strong psh barrier for \( D \) at \( \infty \) and (1) \( \Rightarrow \) (11) is proved.
7. To show that (1) \( \Rightarrow \) (12), we shall use only that \( \mathbb{C} \setminus D_j \) is not pluripolar. We may assume that \( 0 \notin D_j \). Let \( G_j \) be the image of \( D_j \) under the transformation \( z \to 1/z \).
Since $C \setminus G_j$ is not a polar set, there is $\varepsilon > 0$ such that $C \setminus G_j^c$ is not polar, too, where $G_j^c = G_j \cup \varepsilon \mathbb{D}$. Denote by $g_j^c$ the Green function of $G_j^c$. Then $h_j = g_j^c(0; \cdot)$ is a negative harmonic function on $G_j$ with $\lim_{z \to 0} h_j(z) = -\infty$ and $\inf_{G_j \setminus \varepsilon \mathbb{D}} h_j > -\infty$ for any $r > 0$.

Then $\psi_j(z) = h_j(1/z)$ is an antipeak function of $D_j$ at $\infty$ and, hence, $\psi = \sum_{j=1}^{N'} \psi_j$ is an antipeak function for $D$ at $\infty$.

Remarks. (a) A consequence of the fundamental Lempert theorem is the fact that the Carathéodory and Kobayashi distances coincide on any bounded $C^2$-convex domain with $C^2$-boundary (cf. [7] and the references there). Therefore, they coincide on any convex domain.

(b) Assume that $a$ is a regular boundary point of a planar domain $D$, i.e. $a$ admits a local weak subharmonic barrier. Then there is a global strong harmonic barrier at $a$. Indeed we may choose as above a neighbourhood $U$ of $a$ such that $D \setminus U$ admits Green function $g_{D \setminus U}$. Let $\tilde{g} = e^{g_{D \setminus U}(a; \cdot)}$, and $h$ be the associated Perron function

$$h = \sup \{ \tilde{h} \text{ subharmonic on } D : \lim \sup_{z \to b} \tilde{h}(z) \leq \lim \sup_{z \to b} \tilde{g}(z) \quad \forall b \in \partial D \}$$

$$(\infty \in \partial D : \text{ if } D \text{ is unbounded}).$$

Then $h$ is a harmonic function and $\tilde{g} \leq h < 1$. In particular, $\inf_{D \setminus V} h > 0$ for any neighbourhood $V$ of $a$. To see that $-h$ is a strong barrier at $a$, it remains to use that $h(a) = \tilde{g}(a) = 0$ by the continuity of $\tilde{g}$ at the regular point $a$ (cf. Theorem 4.5.1 in [8]).

(c) If a $C$-convex domain $D$ does not verify the equivalent properties of Theorem 2, then it contains a complex line and the holomorphic functions on $D$ can be as bad as the entire functions are. In particular, there are holomorphic functions $f : D \to D$ without fixed points whose sequence of iterates is not compactly divergent. If $D$ is convex, then the converse implication also holds (cf. [2]).

Acknowledgements. The authors thank V. Drensky for his remarks which improved this note.

REFERENCES


Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: nik@math.bas.bg

* Dipartamento di Matematica
Università di Roma “Tor Vergata”
1, Via Della Ricerca Scientifica
00133 Roma, Italy
e-mail: alberto.saracco@gmail.com