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HYPERBOLICITY OF $\mathbb{C}$-CONVEX DOMAINS

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Abstract

We transfer several characterizations of hyperbolic convex domains given in a recent joint paper by Bracci and one of the authors to analogous one for $\mathbb{C}$-convex domains.

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Despite the fact that convexity is not an invariant property in complex analysis, bounded convex domains in $\mathbb{C}^N$ have been intensively studied as prototypes for the general situation.

Natural generalizations of the notion of convexity is $\mathbb{C}$-convexity (cf. [1,5]) which, although non-invariant, reflects the underlying complex vector space structure. A domain $D \subset \mathbb{C}^N$ is called $\mathbb{C}$-convex if any non-empty intersection of D with a complex line is contractible.

We point out the following

Proposition 1 [7]**.** Let $D \subset \mathbb{C}^N$ be a $\mathbb{C}$-convex domain. Then there exist a unique k $(0 \le k \le N)$ and a unique $\mathbb{C}$-convex $D' \subset \mathbb{C}^k$ containing no complex lines, such that up to a linear change of coordinates, $D = D' \times \mathbb{C}^{N-k}$. Moreover, D' is biholomorphic to a bounded domain and it is c-finitely compact (i.e. the balls with respect to the Carathéodory distance of D' are relatively compact in D').

Recall that the Carathéodory pseudodistance is the smallest pseudodistance decreasing under holomorphic mapping and coinciding with the Poincaré distance on the unit disc in $\mathbb{C}$, while the Kobayashi pseudodistance is the largest one with these properties.

Based on Proposition 1, in this note we generalize several characterizations of hyperbolicity obtained in [2] for convex domains to $\mathbb{C}$-convex domains (for the definitions we refer the reader to e.g. [6]).

Theorem 1. Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) $\mathbb{C}$-convex domain. Then the following conditions are equivalent:

(1) D is biholomorphic to a bounded domain;

(2) D is complete with respect to the Carathéodory distance;

(3) D is complete with respect to the Kobayashi distance;

(4) D is Kobayashi hyperbolic;

(5) D admits complete Bergman metric;

(6) D is taut (i.e. the family $\mathcal{O}(\mathbb{D}, D)$ is normal, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc);

(7) D is hyperconvex (i.e. D has a negative plurisubharmonic (psh) exhaustion function);

(8) D is Brody hyperbolic (i.e. D contains no nonconstant entire curves);

(9) D contains no complex lines;

(10) D has N linearly independent separating complex lines (i.e. lines passing through boundary points of D and disjoint from D);

(11) D has a strong psh barrier at ∞ (i.e. a psh function φ such that $\limsup\limits_{z \to a} \varphi(z) < 0 = \lim\limits_{z \to \infty} \varphi(z)$ for any finite $a \in \overline{D}$);

(12) D has an antipeak function at infinity (in sense of Gaussier [4], i.e. a psh function $\varphi < 0$ such that $\liminf_{z \to a} \varphi(z) > -\infty = \lim_{z \to \infty} \varphi(z)$ for any finite $a \in \overline{D}$).

Proof. The implication (i) $\Rightarrow$ (1), $2 \le i \le 12$, trivially follows by Proposition 1 (if (1) does not hold, then D contains a complex line);

(1) $\Rightarrow$ (2) also follows by Proposition 1;

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (8) $\Rightarrow$ (9) trivially hold for any domain;

(2) $\Rightarrow$ (7) is true for any domain, since $\tanh c_D - 1$ is a negative psh exhaustion function of D (here c_D stands for the Carathéodory distance of D);

(7) $\Rightarrow$ (6) and (7) $\Rightarrow$ (5) hold also for any domain; see [6] and [3], respectively;

To show that (1) $\Rightarrow$ (10), (11), (12), we shall use that up to a linear change of coordinates, $D \subset \prod\limits_{j=1}^{N} D_j$, where any D_j is biholomorphic to $\mathbb{D}$ (see [7]). This immediately gives that (1) $\Rightarrow$ (10).

Assume now that $D_1, \dots, D_{N'}$ are unbounded. Since D_j is hyperconvex, it admits a strong subharmonic barrier at any boundary point (including ∞) by Bouligand's lemma for unbounded planar domains (cf. [8]; see also Remark (b)). If φ_j denotes the barrier at ∞, then $\varphi = \max\limits_{1 \le j \le N'} \varphi_j$ is a strong psh barrier for D at ∞ and (1) $\Rightarrow$ (11) is proved.

To show that (1) $\Rightarrow$ (12), we shall use only that $\mathbb{C} \setminus D_j$ is not pluripolar. We may assume that $0 \notin D_j$. Let G_j be the image of D_j under the transformation $z \to 1/z$.

Since $\mathbb{C} \setminus G_j$ is not a polar set, there is $\varepsilon > 0$ such that $\mathbb{C} \setminus G_j^\varepsilon$ is not polar, too, where $G_j^\varepsilon = G_j \cup \varepsilon\mathbb{D}$. Denote by g_j^ε the Green function of G_j^ε. Then $h_j = g_j^\varepsilon(0; \cdot)$ is a negative harmonic function on G_j with $\lim\limits_{z\to 0} h_j(z) = -\infty$ and $\inf\limits_{G_j \setminus r\mathbb{D}} h_j > -\infty$ for any $r > 0$. Then $\psi_j(z) = h_j(1/z)$ is an antipeak function of D_j at ∞ and, hence, $\psi = \sum\limits_{j=1}^{N'} \psi_j$ is an antipeak function for D at ∞. $\diamondsuit$

Remarks. (a) A consequence of the fundamental Lempert theorem is the fact that the Carathéodory and Kobayashi distances coincide on any bounded $\mathbb{C}$-convex domain with C^2-boundary (cf. [7] and the references there). Therefore, they coincide on any convex domain.

(b) Assume that a is a regular boundary point of a planar domain D, i.e. a admits a local weak subharmonic barrier. Then there is a global strong harmonic barrier at a. Indeed we may choose as above a neighbourhood U of a such that $D \cup U$ admits Green function $g_{D\cup U}$. Let $\tilde{g} = e^{g_{D\cup U}(a,\cdot)}$, and h be the associated Perron function

$$h = \sup\{\tilde{h} \text{ subharmonic on } D : \limsup_{z\to b} \tilde{h}(z) \le \limsup_{z\to b} \tilde{g}(z)\ \forall b \in \partial D\}$$

$$(\infty \in \partial D: \text{ if } D \text{ is unbounded}).$$

Then h is a harmonic function and $\tilde{g} \le h < 1$. In particular, $\inf_{D\setminus V} h > 0$ for any neighbourhood V of a. To see that $-h$ is a strong barrier at a, it remains to use that $h(a) = \tilde{g}(a) = 0$ by the continuity of $\tilde{g}$ at the regular point a (cf. Theorem 4.5.1 in [8]).

(c) If a $\mathbb{C}$-convex domain D does not verify the equivalent properties of Theorem 2, then it contains a complex line and the holomorphic functions on D can be as bad as the entire functions are. In particular, there are holomorphic functions $f : D \to D$ without fixed points whose sequence of iterates is not compactly divergent. If D is convex, then the converse implication also holds (cf. [2]).

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