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NON-COMPACT BOUNDARIES OF COMPLEX ANALYTIC VARIETIES

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We treat the boundary problem for complex varieties with isolated singularities, of dimension greater than one, which are contained in a certain class of strongly pseudoconvex, not necessarily bounded open subsets of $\mathbb{C}^n$. We deal with the problem by cutting with a family of complex hyperplanes and applying the classical Harvey–Lawson's theorem for the bounded case [6].

Keywords: Boundary problem; pseudoconvexity; maximally complex submanifold; CR geometry.

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1. Introduction

Let M be a smooth and oriented $(2m+1)$-dimensional real submanifold of some n-dimensional complex manifold X. A natural question arises, whether M is the boundary of an $(m+1)$-dimensional complex analytic subvariety of X. This problem, the so-called *boundary problem*, has been widely treated over the past fifty years when M is compact and X is $\mathbb{C}^n$ or $\mathbb{CP}^n$.

The case when M is a compact, connected curve in $X = \mathbb{C}^n$ $(m = 0)$, has been first solved by Wermer [13] in 1958. In 1966, Stolzenberg [12] proved the same result when M is a union of smooth curves. Later on, in 1975, Harvey and Lawson in [6, 7] solved the boundary problem in $\mathbb{C}^n$ and then in $\mathbb{CP}^n \setminus \mathbb{CP}^r$, in terms of holomorphic chains, for any m. The boundary problem in $\mathbb{CP}^n$ was studied by Dolbeault and Henkin, in [3] for $m = 0$ and in [4] for any m. Moreover, in these two papers the boundary problem is dealt with also for closed submanifolds (with negligible singularities) contained in q-concave (i.e. union of $\mathbb{CP}^q$'s) open subsets of $\mathbb{CP}^n$. This allows M to be non-compact. The results in [3, 4] were extended by Dinh in [2].

The main theorem proved by Harvey and Lawson in [6] is that if $M \subset \mathbb{C}^n$ is compact and maximally complex then M is the boundary of a unique holomorphic chain of finite mass [6, Theorem 8.1]. Moreover, if M is contained in the boundary $b\Omega$ of a strictly pseudoconvex domain Ω then M is the boundary of a complex analytic subvariety of Ω, with isolated singularities [8] (see also [5]). The aim of this paper is to generalize this last result to a non-compact, connected, closed and maximally complex submanifold M of the connected boundary $b\Omega$ of an unbounded weakly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. The pseudoconvexity of Ω is needed both for the local result and to prove that the singularities are isolated.

Maximal complexity of M and extension theorem for CR functions (see [9]) allow us to prove the following semi-global result (see Corollary 3.1). Assume that $n \geq 3$, $m \geq 1$ and the Levi form $\mathcal{L}(b\Omega)$ of $b\Omega$ has at least $n-m$ positive eigenvalues at every point $p \in M$. Then

> there exist a tubular open neighborhood I of $b\Omega$ and a complex submanifold W_0 of $\overline{\Omega} \cap I$ with boundary, such that $bW_0 \cap b\Omega = M$, i.e. a complex manifold $W_0 \subset I \cap \Omega$ such that the closure $\overline{W}_0$ of W_0 in I is a smooth submanifold with boundary M.

A very simple example (see Example 3.1) shows that in general the semi-global result fails to be true for $m = 0$.

In order to prove that W_0 extends to a complex analytic subvariety W of Ω with boundary M we first treat the case when Ω is convex and does not contain straight lines. This is the crucial step. For technical reasons we divide the proof in two cases: $m \geq 2$ and $m = 1$. We cut $\overline{\Omega}$ by a family of real hyperplanes H_λ which intersect M along smooth compact submanifolds. Then the natural foliation on each H_λ by complex hyperplanes induces on $M \cap H_\lambda$ a foliation by compact maximally complex $(2m-1)$-dimensional real manifolds M'. Thus a natural way to proceed is to apply Harvey–Lawson's theorem to each M' and to show that the family $\{W'\}$ of the corresponding Harvey–Lawson solutions actually organizes in a complex analytic subvariety W, giving the desired extension (see Theorem 4.1). This is done by following an idea of Zaitsev (see Lemma 4.1).

The same method of proof is used in the last section in order to treat the problem when Ω is pseudoconvex. In this case, M is requested to fulfill an additional condition. Precisely,

($\star$) if $\overline{M}^\infty$ denotes the closure of $M \subset \mathbb{C}^n \subset \mathbb{CP}^n$ in $\mathbb{CP}^n$, then there exists an algebraic hypersurface V such that $V \cap \overline{M}^\infty = \emptyset$.

Equivalently

($\star'$) there exists a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$M \subset \left\{z \in \mathbb{C}^n : |P(z)|^2 > (1+|z|^2)^{\deg P}\right\}.$$

A similar condition was first pointed out by Lupacciolu [10] in studying the extension problem for CR functions in unbounded domains. It allows us to build a nice family of hypersurfaces, which play the role of the hyperplanes in the convex case, and so to prove the main theorem of the paper:

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$ $(n \geq 3)$ with smooth boundary $b\Omega$ and M be a maximally complex closed $(2m+1)$-dimensional real submanifold $(m \geq 1)$ of $b\Omega$. Assume that*

(i) *$b\Omega$ is weakly pseudoconvex and the Levi form $\mathcal{L}(b\Omega)$ has at least $n-m$ positive eigenvalues at every point of M;*
(ii) *M satisfies condition $(\star)$.*

Then there exists a unique $(m+1)$-dimensional complex analytic subvariety W of Ω, such that $bW = M$. Moreover the singular locus of W is discrete and the closure of W in $\overline{\Omega} \setminus \mathsf{Sing}\,(W)$ is a smooth submanifold with boundary M.

We remark that if Ω is bounded, the result is contained in the papers by Harvey and Lawson [6–8]. Here we treat the case when Ω is unbounded.

We do not deal with the 1-dimensional case. There are two different kinds of difficulties. First of all, a semi-global strip as in Corollary 3.1 may not exist (see Example 3.1). Secondly, even though it does exist, it could be non-extendable to the whole Ω (see Example 4.1) and it is not clear at all how it is possible to generalize the *moments condition* (see [6]).

Another similar approach can be followed to treat the *semi-local* boundary problem, i.e. given an open subset U of the boundary of Ω, find an open subset $\Omega' \subset \Omega$ such that, for any maximally complex submanifold $M \subset U$, there exists a complex subvariety W of Ω' whose boundary is M. We deal with this problem in a work in preparation (see [1]).

2. Definitions and Notations

We briefly recall some well known notions of CR geometry that will be used in the paper.

Let $N \subset \mathbb{C}^n$ be a smooth connected real submanifold, and let $p \in N$. We denote by $T_p(N)$ the tangent space of N at the point p, and by $H_p(N)$ the holomorphic tangent space of N at the point p.

A $(2k+1)$-dimensional real submanifold $N \subset \mathbb{C}^n$, $k \geq 1$, is said to be a CR *submanifold* if $\mathsf{dim}_{\mathbb{C}} H_p(N)$ is constant along N. When this is the case, $H(N) = \cup_p H_p(N)$ is a subbundle of the tangent bundle $T(N)$. If $\mathsf{dim}_{\mathbb{C}} H_p(N)$ is the greatest possible, i.e. $\mathsf{dim}_{\mathbb{C}} H_p(N) = k$ for every p, N is said to be *maximally complex*.

A C^∞ function $f : N \to \mathbb{C}$ is said to be a CR *function* if for a C^∞ extension (and hence for any) $\widetilde{f} : U \to \mathbb{C}$ (U being a neighborhood of N) we have

$$(\overline{\partial}\widetilde{f})|_{H(N)} = 0. \tag{2.1}$$

In particular the restriction of a holomorphic function to a CR submanifold is a CR function. It is immediately seen that f is CR if and only if

$$df \wedge (dz_1 \wedge \cdots \wedge dz_n)|_N = 0. \tag{2.2}$$

Similarly N is maximally complex if and only if

$$(dz_{j_1} \wedge \cdots \wedge dz_{j_{k+1}})|_N = 0,$$

for any $(j_1, \ldots, j_{k+1}) \in \{1, \ldots, n\}^{k+1}$.

Finally we observe that the boundary M of a complex submanifold W with $\mathsf{dim}_{\mathbb{C}} W > 1$ is maximally complex. Indeed, for any $p \in bW = M$, $T_p(bW)$ is a real hyperplane of $T_p(W) = H_p(W)$ and so is $J(T_p(bW))$. Hence $H_p(bW) = T_p(bW) \cap J(T_p(bW))$ is of real codimension 2 in $H_p(W)$.

If $\mathsf{dim}_{\mathbb{C}} W = 1$ and bW is compact then for any holomorphic $(1,0)$-form ω we have

$$\int_M \omega = \iint_W d\omega = \iint_W \partial\omega = 0,$$

since $\partial\omega|_W \equiv 0$. This condition for M is called *moments condition* (see [6]).

By the same arguments, a $(2n-1)$-dimensional real submanifold of $\mathbb{C}^n$ is maximally complex.

3. The Local and Semi-Global Results

The aim of this section is to prove the local and the semi-global results. Given a smooth real hypersurface S in $\mathbb{C}^n$, we denote by $\mathcal{L}_p(S)$ the Levi form of S at the point p. Let 0 be a point of M. We have the following inclusions of tangent spaces:

$$\mathbb{C}^n \supset T_0(S) \supset H_0(S) \supset H_0(M);$$
$$T_0(S) \supset T_0(M) \supset H_0(M).$$

Lemma 3.1. *Let M be a maximally complex submanifold of a smooth real hypersurface S, $\mathsf{dim}_{\mathbb{R}} M = 2m+1$, $m \geq 1$, $0 \in M$. Suppose that $\mathcal{L}_0(S)$ has at least $n-m$ eigenvalues of the same sign. Then*

$$H_0(S) \not\supset T_0(M).$$

Proof. Should the thesis fail we would have the following chain of inclusions

$$\mathbb{C}^n \supset T_0(S) \supset H_0(S) \supset T \supset T_0(M) \supset H_0(M),$$

where T is the smallest complex space containing $T_0(M)$ (since M is maximally complex, $\mathsf{dim}_{\mathbb{C}} T = m+1$). Hence, we may choose in a neighborhood of 0 local complex coordinates $z_k = x_k + iy_k$, $k = 1, \ldots, m+1$, $w_l = u_l + iv_l$, $l = m+2, \ldots, n$, in such a way that:

- $H_0(M) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k)$, $k = 1, \ldots, m$;
- $T_0(M) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial x_{m+1})$, $k = 1, \ldots, m$;

- $T = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k)$, $k = 1, \dots, m+1$;
- $H_0(S) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_l, \partial/\partial v_l)$, $k = 1, \dots, m+1$, $l = m+2, \dots, n-1$, if $m + 2 \le n - 1$;
  or
- $H_0(S) = T$, if $m = n - 2$;
- $T_0(S) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_l, \partial/\partial v_l, \partial/\partial u_n)$, $k = 1, \dots, m+1$, $l = m+2, \dots, n-1$, if $m + 2 \le n - 1$;
  or
- $T_0(S) = \mathsf{span}\ (\partial/\partial x_k, \partial/\partial y_k, \partial/\partial u_n)$ $k = 1, \dots, m+1$, if $m = n - 2$.

We denote by z the first $m + 1$ coordinates, by $\hat{z}$ the first m, and by π the projection on T; π is obviously a local embedding of M near 0, and we set $M_0 = \pi(M)$.

Locally at 0, S is a graph over its tangent space:

$$S = \{v_n = h(u_n, u_j, v_j, x_i, y_i)\}.$$

Observe that the Levi form of h has $n - m$ eigenvalues of the same sign. In order to obtain a similar description of M, we proceed as follows. First, we have

$$M_0 = \{(\hat{z}, z_{m+1}) : y_{m+1} = \varphi(\hat{z}, x_{m+1})\}.$$

Then, we choose $f_j(\hat{z}, x_{m+1}) = f_j^1(\hat{z}, x_{m+1}) + i f_j^2(\hat{z}, x_{m+1})$ (where f_j^1 and f_j^2 are real-valued) defined in a neighborhood of M_0 in T in such a way that

$$M = \{w_{m+2} = f_{m+2}(\hat{z}, x_{m+1}), \dots, w_n = f_n(\hat{z}, x_{m+1})\}.$$

Observe that the function $(f_{m+2}(\hat{z}, x_{m+1}), \dots, f_n(\hat{z}, x_{m+1}))$ is just $\pi^{-1}|_{M_0}$, and since M is maximally complex it has to be a CR map.

By hypothesis, the following equation holds in a neighborhood of 0:

$$f_n^2(\hat{z}, x_{m+1}) = h\left(f_n^1(\hat{z}, x_{m+1}), f_j^k(\hat{z}, x_{m+1}), \hat{z}, x_{m+1}\right).$$

After a computation of the second derivatives, taking into account that all first derivatives of f_j^k, of h and of φ vanish in the origin, we obtain

$$\frac{\partial^2 f_n^2}{\partial z_j \partial \overline{z}_k}(0) = \frac{\partial^2 h}{\partial z_j \partial \overline{z}_k}(0),$$

i.e. the Levi form of h and f_n^2 coincide in $H_0(M)$. By hypothesis $\mathcal{L}_0(h)$ is strictly positive definite on a non-zero subspace of $H_0(M)$. We shall obtain a contradiction by showing that $\mathcal{L}_0(f_n)$ (and hence $\mathcal{L}_0(f_n^2)$) vanishes on $H_0(M)$. Let $\xi \in H_0(M)$. We may assume (up to unitary linear transformation of coordinates of $H_0(M)$) that $\xi = \partial/\partial z_1$.

Set $f \doteqdot f_n$. Then, since f is a CR function on M_0, we have:

$$\frac{\partial}{\partial \overline{z}_k} f(\hat{z}, x_{m+1}) = -\alpha(\hat{z}, x_{m+1}) \frac{\partial}{\partial \overline{z}_k} \varphi(\hat{z}, x_{m+1}), \quad k = 1, \dots, m$$

and

$$\frac{\partial}{\partial \overline{z}_{m+1}} f(\hat{z}, x_{m+1}) = -i\alpha(\hat{z}, x_{m+1}) + \alpha(\hat{z}, x_{m+1}) \frac{\partial}{\partial x_{m+1}} \varphi(\hat{z}, x_{m+1}),$$

where $\alpha(\hat{z}, x_{m+1})$ is a complex valued function. Differentiating and calculating in 0, we obtain

$$\frac{\partial^2 f}{\partial z_1 \partial \overline{z_1}}(0) = \alpha(0) \frac{\partial^2 \varphi}{\partial z_1 \partial \overline{z}_1}(0), \tag{3.1}$$

$$0 = \frac{\partial f}{\partial x_{m+1}}(0) = i\alpha(0), \tag{3.2}$$

i.e. $\alpha(0) = 0$. From (3.1) we deduce that $\partial^2 f / \partial z_1 \partial \overline{z}_1(0) = 0$. Contradiction. □

Lemma 3.2. *Under the hypothesis of Lemma* 3.1, *assume that S is the boundary of an unbounded domain $\Omega \subset \mathbb{C}^n$, $0 \in M$ and that the Levi form of S has at least $n - m$ positive eigenvalues. Then*

(i) *there exists an open neighborhood U of* 0 *and an $(m+1)$-dimensional complex submanifold $W_0 \subset U$ with boundary, such that $bW_0 = M \cap U$;*
(ii) *$W_0 \subset \overline{\Omega} \cap U$.*

Proof. To prove the first assertion, observe that to obtain $\mathcal{L}_0^M(\zeta_0, \overline{\zeta}_0)$ it suffices to choose a smooth local section ζ of $H_0(M)$ such that $\zeta(0) = \zeta_0$ and compute the projection of the bracket $[\zeta, \overline{\zeta}](0)$ on the real part of $T_0(M)$. By hypothesis, the intersection of the space where $\mathcal{L}_0(S)$ is positive with $H_0(M)$ is non-empty; take η_0 in this intersection. Then $\mathcal{L}_0^M(\eta_0, \overline{\eta}_0) \neq 0$. Suppose, by contradiction, that the bracket $[\eta, \overline{\eta}](0)$ lies in $H_0(M)$, i.e. its projection on the real part of the tangent of M is zero. Then, if $\widetilde{\eta}$ is a local smooth extension of the field η to S, we have $[\widetilde{\eta}, \overline{\widetilde{\eta}}](0) = [\eta, \overline{\eta}](0) \in H_0(M)$. Since $H_0(M) \subset H_0(S)$, this would mean that the Levi form of S in 0 is zero in η_0. Now, we project (generically) M over a $\mathbb{C}^{m+1}$ in such a way that the projection π is a local embedding near 0: since the restriction of π to M is a CR function, and since the Levi form of M has — by the arguments stated above — at least one positive eigenvalue, it follows that the Levi form of $\pi(M)$ has at least one positive eigenvalue. Thus, in order to obtain W_0, it is sufficient to apply the Lewy extension theorem [9] to the CR function $\pi^{-1}|_M$.

As for the second statement, we observe that the projection by π of the normal vector of S pointing towards Ω lies into the domain of $\mathbb{C}^{m+1}$ where the above extension W_0 is defined. Indeed, the extension result in [9] gives a holomorphic function in the connected component of (a neighborhood of 0 in) $\mathbb{C}^n \setminus \pi(M)$ for which $\mathcal{L}_0(\pi(M))$ has a positive eigenvalue when $\pi(M)$ is oriented as the boundary of this component. This is precisely the component towards which the projection of the normal vector of S points when the orientations of S and M are chosen accordingly. This fact, combined with Lemma 3.1 (which states that any extension of M must be transverse to S) implies that locally $W_0 \subset \Omega \cap U$. □

Corollary 3.1 (Semi-Global Existence of W). *Under the same hypothesis of Lemma* 3.2, *there exist an open tubular neighborhood I of S in $\overline{\Omega}$ and an $(m+1)$-dimensional complex submanifold W_0 of $\overline{\Omega} \cap I$, with boundary, such that $S \cap bW_0 = M$.*

Proof. By Lemma 3.2, for each point $p \in M$, there exist a neighborhood U_p of p and a complex manifold $W_p \subset \overline{\Omega} \cap U_p$ bounded by M. We cover M with countable many such open sets U_i, and consider the union $W_0 = \cup_i W_i$. W_0 is contained in the union of the U_i's, hence we may restrict it to a tubular neighborhood I_M of M. It is easy to extend I_M to a tubular neighborhood I of S.

The fact that $W_i|_{U_{ij}} = W_j|_{U_{ij}}$ if $U_i \cap U_j = U_{ij} \neq \emptyset$ immediately follows from the construction made in Lemma 3.2, in view of the uniqueness of the holomorphic extension of CR functions. □

Example 3.1. Corollary 3.1 could be restated by saying that if a submanifold $M \subset S$ ($\mathsf{dim}_{\mathbb{R}} M \geq 3$) is locally extendable at each point as a complex manifold, then (one side of) the extension lies in Ω. This is no longer true, in general, for curves, as shown in $\mathbb{C}^n_{(z_1,\ldots,z_{n-1},w)}$, $z_k = x_k + iy_k$, $w = u + iv$, by the following case:

$$S = \left\{ v = u^2 + \sum_k |z_k|^2 \right\}, \quad \Omega = \left\{ v > u^2 + \sum_k |z_k|^2 \right\},$$
$$M = \left\{ y_1 = 0,\ v = x_1^2, u = 0, z_2 = \cdots = z_{n-1} = 0 \right\}$$

and

$$W = \left\{ w = iz_1^2, z_2 = \cdots = z_{n-1} = 0 \right\};$$

we have that $S \cap W = M$ and $W \subset \mathbb{C}^n \setminus \Omega$.

Remark 3.1. Suppose that S is strongly pseudoconvex and choose, in $\mathbb{C}^n_{(z_1,\ldots,z_n)}$, a local strongly plurisubharmonic equation ρ for S: $S = \{\rho = 0\}$. Consider the curve

$$\gamma = \{z_j = \gamma_j(t),\ j = 1,\ldots,n,\ t \in (-\varepsilon, \varepsilon)\} \subset S.$$

Assume that γ is real analytic, so that locally there exists a complex extension $\widetilde{\gamma} \supset \gamma$. Then one side of $\widetilde{\gamma}$ lies in Ω if and only if

$$\sum_j \mathrm{Re} \frac{\partial \rho}{\partial z_j} \frac{\partial \gamma_j}{\partial t} \neq 0. \tag{3.3}$$

Observe that condition (3.3), which depends only on γ (when S is given), is not satisfied in Example 3.1. Sufficiency of (3.3) is true when S is *any* real hypersurface: indeed, from a geometric point of view, the condition is equivalent to the transversality of $T(\widetilde{\gamma})$ and $H(S)$ (and hence $T(S)$). Pseudoconvexity is required to establish the necessity.

4. The Global Result

In order to make the proof more transparent we first treat the case when Ω is an unbounded convex domain with smooth boundary $b\Omega$. In the next section, we will prove the main theorem in all its generality.

Theorem 4.1. *Let M be a maximally complex (connected) $(2m+1)$-dimensional real submanifold $(m \geq 1)$ of $b\Omega$. Assume that Ω is convex and does not contain straight lines and $b\Omega = S$ satisfies the conditions of Lemma 3.1. Then there exists an $(m+1)$-dimensional complex subvariety W of Ω, with isolated singularities, such that $bW = M$.*

We observe that under the hypothesis of Theorem 4.1, there exists a complex strip in a tubular neighborhood with boundary M (see Corollary 3.1). Moreover, since Ω does not contain straight lines, we can approximate uniformly from both sides $b\Omega$ by strictly convex domains, see [11]. It follows that we can find a real hyperplane L such that, for any translation L' of L, $L' \cap \overline{\Omega}$ is a compact set. We choose an exhaustive sequence L_k of such hyperplanes, and we set Ω_k as the bounded connected component of $\Omega \setminus L_k$. Then, approximating from inside, we can choose a strictly convex open subset $\Omega'_k \subset \Omega$ such that $b\Omega'_k \cap \Omega_k \subset I$, where I is the tubular neighborhood of Corollary 3.1. It is easily seen, then, that we are in the situation of the following.

Proposition 4.2. *Let $D \Subset B \Subset \mathbb{C}^n$ $(n \geq 4)$ be two strictly convex domains. Let $D_+ = D \cap \{\mathsf{Re}\, z_n > 0\}$, $B_+ = B \cap \{\mathsf{Re}\, z_n > 0\}$. Then every $(m+1)$-dimensional complex subvariety $(m \geq 2)$ with isolated singularities, $A \subset B_+ \setminus \overline{D}_+ \doteqdot C_+$, is the restriction of a complex subvariety $\widetilde{A}$ of B_+ with isolated singularities.*

We treat the cases $m \geq 2$ and $m = 1$ separately. Indeed all the main ideas of the proof lie in the case $m \geq 2$, while the case $m = 1$ simply adds technical difficulties.

4.1. *M is of dimension at least* 5: $m \geq 2$

Before proving Proposition 4.2, we make some considerations and we prove two lemmata that will be useful.

Let φ be a strictly convex function[a] defined in a neighborhood of B such that $B = \{\varphi < 0\}$. Fixing $\varepsilon > 0$ small enough, $B' = \{\varphi < -\varepsilon\}$ is a strictly convex domain of B whose boundary H intersects A in a smooth maximally complex submanifold N. A natural way to proceed is to slice N with complex hyperplanes, in order to apply Harvey–Lawson's theorem. Each slice of B' is strictly convex, hence strongly pseudoconvex, and so the holomorphic chain we obtain is contained in B'. Thus the set made up by collecting the chains is contained in B'. Analyticity of this set is the hard part of the proof.

[a]In the general case φ will be a strongly plurisubharmonic function.

Because of Sard's lemma, for all $z \in D_+$, there exist a vector v arbitrarily close to $\partial/\partial z_n$, and $k \in \mathbb{C}$ such that $z \in v_k \doteqdot v^\perp + k$ and $A_k \doteqdot v_k \cap N$ is transversal and compact, and thus smooth.

In a neighborhood of each fixed $z_0 \in D_+$, the same vector v realizes the transversality condition. Hence we should now fix our attention to a neighborhood of the form $\widehat{U} \doteqdot \bigcup_{k\in U} v_k \cap B_+$, where v_{k_0} is the vector corresponding to z_0 and $U \subset \mathbb{C}$ a neighborhood of k_0.

Let $\pi : \widehat{U} \to \mathbb{C}^m$ be a generic projection: we use (w', w) as holomorphic coordinates on $v_{k_0} = \mathbb{C}^m \times \mathbb{C}^{n-m-1}$ (and also for k near to k_0). Let $V_k = \mathbb{C}^m \setminus \pi(A_k)$, and $V = \cap_k V_k$.

Since A_{k_0} has a local extension (given by $v_{k_0} \cap A$), it is maximally complex and so, by Harvey–Lawson's theorem, there is a holomorphic chain $\widetilde{A}_{k_0}$ with $b\widetilde{A}_{k_0} = A_{k_0}$, which extends holomorphically A_{k_0}.

Our goal is to show that $\widetilde{A}_U = \cup_k \widetilde{A}_k$ is analytic in $\pi^{-1}(V)$. From this, it will follow that $\widetilde{A}_U$ is an analytic subvariety of $\widehat{U}$, π being a generic projection.

Following an idea of Zaitsev, for $k \in U$, $w' \in \mathbb{C}^m \setminus \pi(A_k)$ and $\alpha \in \mathbb{N}^{n-m-1}$, we define

$$I^\alpha(w', k) \doteqdot \int_{(\eta',\eta)\in A_k} \eta^\alpha \omega_{BM}(\eta' - w'),$$

ω_{BM} being the Bochner–Martinelli kernel.

Lemma 4.1 (Zaitsev). *Let $F(w', k)$ be the multiple-valued function which represents $\widetilde{A}_k$ on $\mathbb{C}^m \setminus \pi(A_k)$; then, if we denote by $P^\alpha(F(w', k))$ the sum of the αth powers of the values of $F(w', k)$, the following holds*:

$$P^\alpha(F(w', k)) = I^\alpha(w', k).$$

In particular, $F(w', k)$ is finite.

Proof. Let V_0 be the unbounded component of V_k (where, of course, $P^\alpha(F(w', k)) = 0$). It is easy to show, following [6], that on V_0 also $I^\alpha(F(w', k)) = 0$: in fact, if w' is far enough from $\pi(A_k)$, then $\beta = \eta^\alpha \omega_{BM}(\eta' - w')$ is a regular $(m, m-1)$-form on some Stein neighborhood O of A_k. So, since in O there exists γ such that $\overline{\partial}\gamma = \beta$, we may write in the language of currents

$$[A_k](\beta) = [A_k]_{m,m-1}(\overline{\partial}\gamma) = \overline{\partial}[A_k]_{m,m-1}(\gamma) = 0.$$

In fact, since A_k is maximally complex, $[A_k] = [A_k]_{m,m-1} + [A_k]_{m-1,m}$ and $\overline{\partial}[A_k]_{m,m-1} = 0$, see [6]. Moreover, since $[A_k](\beta)$ is analytic in the variable w', $[A_k](\beta) = 0$ for all $w' \in V_0$.

To conclude our proof, we just need to show that the "jumps" of the functions $P^\alpha(F(w', k))$ and $I^\alpha(w', k)$ across the regular part of the common boundary of two components of V_k are the same.

So, let $z' \in \pi(A_k)$ be a regular point in the common boundary of V_1 and V_2. Locally in a neighborhood of z', we can write $\widetilde{A}_k$ as a finite union of graphs of holomorphic functions, whose boundaries A_k^i are either in A_k or empty. In the first case, the A_k^i are CR graphs over $\pi(A_k)$ in the neighborhood of z'. We may thus consider the jump j_i of $P^\alpha(F(w',k))$ due to a single function. We remark that the jump for a function f is $j_i = f(z')^\alpha$. The total jump will be the sum of them.

To deal with the jump of $I^\alpha(w',k)$ across z', we split the integration set in the sets A_k^i (thus obtaining the integrals I_i^α) and $A_k \setminus \cup_i A_k^i$ (I_0^α). Thanks to Plemelj's formulas (see [6]), the jumps of I_i^α are precisely j_i. Moreover, since the form $\eta^\alpha \omega_{BM}(\eta' - z')$ is C^∞ in a neighborhood of $A_k \setminus \cup_i A_k^i$, the jump of I_0^α is 0. So $P^\alpha(F(w',k)) = I^\alpha(w',k)$. $\square$

Remark 4.1. Lemma 4.1 implies, in particular, that the functions $P^\alpha(F(w',k))$ are continuous in k. Indeed, they are represented as integrals of a fixed form over submanifolds A_k which vary continuously with the parameter k.

The functions $P^\alpha(F(w',k))$ and the holomorphic chain $\widetilde{A}_k$ uniquely determine each other and so, proving that the union over k of the $\widetilde{A}_k$ is an analytic set is equivalent to proving that the functions $P^\alpha(F(w',k))$ are holomorphic in the variable $k \in U \subset \mathbb{C}$.

Lemma 4.2. *$P^\alpha(F(w',k))$ is holomorphic in the variable $k \in U \subset \mathbb{C}$, for each $\alpha \in \mathbb{N}^{n-m-1}$.*

Proof. The proof is very similar to the one of Lewy's main lemma in [9]. Let us fix a point $(w', \underline{k})$ such that $w' \notin A_{\underline{k}}$ (this condition remains true for $k \in B_\epsilon(\underline{k})$). Consider as domain of $P^\alpha(F)$ the set $\{w'\} \times B_\epsilon(\underline{k})$. In view of Morera's theorem, we need to prove that for any simple curve $\gamma \subset B_\epsilon(\underline{k})$,

$$\int_\gamma P^\alpha(F(w',k))dk = 0.$$

Let $\Gamma \subset B_\epsilon(\underline{k})$ be an open set such that $b\Gamma = \gamma$. By $\gamma * A_k$ ($\Gamma * A_k$) we mean the union of A_k along γ (along Γ). Note that these sets are submanifolds of N ($\Gamma * A_k$ is an open subset) and $b(\Gamma * A_k) = \gamma * A_k$. By Lemma 4.1 and Stoke's theorem

$$\begin{aligned}
\int_\gamma P^\alpha(F(w',k))dk &= \int_\gamma I^\alpha(w',k)dk \\
&= \int_\gamma \left(\int_{(\eta',\eta)\in A_k} \eta^\alpha \omega_{BM}(\eta' - w') \right) dk \\
&= \iint_{\gamma * A_k} \eta^\alpha \omega_{BM}(\eta' - w') \wedge dk \\
&= \iint_{\Gamma * A_k} d\left(\eta^\alpha \omega_{BM}(\eta' - w') \wedge dk\right) \\
&= \iint_{\Gamma * A_k} d\eta^\alpha \wedge \omega_{BM}(\eta' - w') \wedge dk \\
&= 0.
\end{aligned}$$

The last equality follows from the fact that since η^α is holomorphic, only holomorphic differentials appear in $d\eta^\alpha$. Since all the holomorphic differentials supported by $\Gamma * A_k$ already appear in $\omega_{BM}(\eta' - w') \wedge dk$, the integral is zero. □

We may now prove Proposition 4.2.

Proof of Proposition 4.2 ($m \geq 2$). Up to this point we have extended the complex manifold A to an analytic set

$$\widetilde{A}_U \doteqdot A \cup \bigcup_{k \in U} \widetilde{A}_k \subset V_U \doteqdot C_+ \cup \bigcup_{k \in U} (v_k \cap B_+) .$$

The open sets V_U are an open covering of B_+.

Moreover the open sets $\omega_U \doteqdot \bigcup_{k \in U}(v_k \cap B_+)$ are an open covering of each compact set $K_\delta \doteqdot \overline{B}' \cap \{\mathsf{Re}\, z_n \geq \delta\}$. Hence there exist $\omega_1, \ldots, \omega_l$ which cover K_δ and such that $\omega_i \cap \omega_{i+1} \cap C_+ \neq \emptyset$, for $i = 1, \ldots, l-1$ and therefore there exists a countable open cover $\{\omega_i\}_{i \in \mathbb{N}}$ of $\overline{B}' \cap B_+$ such that, for all $i \in \mathbb{N}$, $\omega_i \cap \omega_{i+1} \cap C_+ \neq \emptyset$.

So we may extend A to $C_+ \cup \omega_1$ by proceeding as above.

Suppose now that we have extended A to $C^i \doteqdot C_+ \cup \bigcup_{j=1}^{i} \omega_j$ with an analytic set A_i. On the non-empty intersection $C^i \cap \omega_{i+1} \cap C_+$ A_i and the extension $\widetilde{A}_{i+1}$ of A to $C_+ \cup \omega_{i+1}$ coincide (as they both coincide with A), hence by analiticity they coincide everywhere. Consequently we may extend A to C^{i+1} by $A_{i+1} \doteqdot A_i \cup \widetilde{A}_{i+1}$. It follows that, defining

$$\widetilde{A} \ \doteqdot\ A \cup \bigcup_{j \in \mathbb{N}} A_j,$$

$\widetilde{A}$ is the desired extension of A to B_+. In order to conclude the proof we have to show that $\widetilde{A}$ has isolated singularities. Let $\mathsf{Sing}\ (\widetilde{A}) \subset B'_+$ be the singular locus of $\widetilde{A}$.

Recall that φ is a strictly convex defining function for B. Let us consider the family

$$(\phi_\lambda = \lambda\varphi + (1-\lambda)\mathsf{Re}\, z_n)_{\lambda \in [0,1]}$$

of strictly convex functions. For λ near to 1, $\{\phi_\lambda = 0\}$ does not intersect the singular locus $\mathsf{Sing}\ (\widetilde{A})$. Let $\overline{\lambda}$ be the biggest value of λ for which $\{\phi_\lambda = 0\} \cap \mathsf{Sing}\ (\widetilde{A}) \neq \emptyset$. Then

$$\left\{\phi_{\overline{\lambda}} < 0\right\} \cap B_+ \subset B_+$$

is a Stein domain in whose closure the analytic set $\mathsf{Sing}\ (\widetilde{A})$ is contained, touching the boundary in a point of strict convexity. So, by Kontinuitätsatz,

$$\{\phi_{\overline{\lambda}} = 0\} \cap \mathsf{Sing}\ (\widetilde{A})$$

is a set of isolated points in $\mathsf{Sing}\ (\widetilde{A})$. By repeating the argument, we conclude that $\mathsf{Sing}\ (\widetilde{A})$ is made up by isolated points. □

Proof of Theorem 4.1 ($m \geq 2$). Thanks to Corollary 3.1, we have a regular submanifold W_1 of a tubular neighborhood I, with boundary M.

Suppose $0 \in M$. The real hyperplanes $H_k \doteqdot T_0(S) + k$, $k \in \mathbb{R}$, intesect S in a compact set. If the intersection is non-empty, $\overline{\Omega}$ is divided in two sets. Let Ω_k be the relatively compact one. We can choose a sequence H_{k_n} such that Ω_{k_n} is an exaustive sequence for $\overline{\Omega}$.

We apply Proposition 4.2 with $B_+ = \Omega_{k_n}$, $C_+ = I \cap \Omega_{k_n}$, and $A = W_1 \cap \Omega_{k_n}$, to obtain an extension of W_1 in Ω_{k_n}. Since, by the identity principle, two such extensions coincide in $\Omega_{k_{\min\{n,m\}}}$, their union is the desired submanifold W. $\square$

4.2. *M is of dimension* 3: $m = 1$

We prove now the statement of Proposition 4.2 for $m = 1$.

Our first step is to show that when we slice transversally N with complex hyperplanes, we obtain 1-dimensional real submanifolds which satisfy the moments condition.

Again, we fix our attention to a neighborhood of the form $\widehat{U} \doteqdot \bigcup_{k \in U} v_k \cap B_+$. In $\widehat{U}$, with coordinates $w_1, \ldots, w_{n-1}, k$, we choose an arbitrary holomorphic $(1,0)$-form Ω which is constant with respect to k.

Lemma 4.3. *The function*

$$\Phi_\omega(k) = \int_{A_k} \omega$$

is holomorphic in U.

Proof. We use again Morera's theorem. We need to prove that for any simple curve $\gamma \subset U$, $\gamma = b\Gamma$,

$$\int_\gamma \Phi_\omega(k) dk = 0.$$

Applying Stoke's theorem, we have

$$\begin{aligned}
\int_\gamma \Phi_\omega(k) dk &= \int_\gamma \left(\int_{A_k} \omega \right) dk \\
&= \iint_{\gamma * A_k} \omega \wedge dk \\
&= \iint_{\Gamma * A_k} d(\omega \wedge dk) \\
&= \iint_{\Gamma * A_k} \partial\omega \wedge dk \\
&= 0.
\end{aligned}$$

The last equality is due to the fact that $\Gamma * A_k \subset N$ is maximally complex and thus supports only $(2,1)$ and $(1,2)$-forms, while $\partial\omega \wedge dk$ is a $(3,0)$-form. $\square$

Now we can prove Proposition 4.2 and Theorem 4.1 also when $m = 1$.

We can find a countable covering of B_+ made of open subsets $\omega_i = \widehat{U}_i \cap B_+$ in such a way that:

(1) $\omega_0 \subset C_+$;
(2) if

$$B_l = \bigcup_{i=1}^{l} \omega_i,$$

then $\omega_{l+1} \cap B_l \supset v_{l+1} \cap B_+$, where v_{l+1} is a complex hyperplane in $\widehat{U}_{l+1}$.

Now, suppose we have already found $\widetilde{A}_l$ that extends A on B_l (observe that in $B_0 = \omega_0$, $\widetilde{A}_0 = A$). To conclude the proof we have to find $\widetilde{A}_{l+1}$ extending A on B_{l+1}.

Each slice of N in B_l is maximally complex, and so are $v_{l+1} \cap N$ and $v_\epsilon \cap N$, for $v_\epsilon \subset \omega_{l+1}$ sufficiently near to v_{l+1} (because they are in B_l as well).

Now we use Lemma 4.3 with $\widehat{U} = \widehat{U}_{l+1}$. What we have just observed implies that, for all holomorphic $(1,0)$-form η, $\Phi_\eta(k)$ vanishes in an open subset of U and so is identically zero on U. This implies that all slices in ω_{l+1} are maximally complex. Again we may apply Harvey–Lawson's theorem slice by slice and conclude by the methods of Proposition 4.2.

4.3. *M is of dimension* 1: *m* = 0

We have already observed that if M is one-dimensional the local extension inside Ω may not exist (see Example 3.1). Even though there is a local strip in which we have an extension, the methods used to prove Proposition 4.2 do not work, since the transversal slices M are either empty or isolated points. Indeed, as the following example shows, that extension result does not hold for $m = 0$.

Example 4.1. Using the notation of Proposition 4.2, in $\mathbb{C}^2$ let B and D be the balls

$$B = \{|z_1|^2 + |z_2|^2 < c\}, \quad D = \{|z_1|^2 + |z_2|^2 < \varepsilon\}, \quad c > \varepsilon > 2.$$

Consider the connected irreducible analytic set of codimension one

$$A = \{(z_1, z_2) \in B_+ \ : \ z_1 z_2 = 1\}$$

and its restriction A_C to C_+. If A_C has two connected components, A_1 and A_2, when we try to extend A_1 (analytic set of codimension one on C_+) to B_+, its restriction to C_+ will contain also A_2. So A_1 is an analytic set of codimension one on C_+ that does not extend on B_+.

So, let us prove that A_C has indeed two connected components. A point of A (of A_C) can be written as $z_1 = \rho e^{i\theta}$, $z_2 = \frac{1}{\rho}e^{-i\theta}$, with $\rho \in \mathbb{R}^+$ and $\theta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. Hence, points in A_C satisfy

$$2 < \varepsilon < \rho^2 + \frac{1}{\rho^2} < c \;\Rightarrow\; 2 < \sqrt{\varepsilon + 2} < \rho + \frac{1}{\rho} < \sqrt{c+2}.$$

Since $f(\rho) = \rho + 1/\rho$ is monotone decreasing up to $\rho = 1$ (where $f(1) = 2$), and then monotone increasing, there exist a and b such that the inequalities are satisfied when $a < \rho < b < 1$, or when $1 < 1/b < \rho < 1/a$. A_C is thus the union of the two disjoint open sets

$$A_1 = \left\{\left(\rho e^{i\theta}, \frac{1}{\rho}e^{-i\theta}\right) \in \mathbb{C}^2 \;\middle|\; a < \rho < b, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\};$$

$$A_2 = \left\{\left(\rho e^{i\theta}, \frac{1}{\rho}e^{-i\theta}\right) \in \mathbb{C}^2 \;\middle|\; a < \frac{1}{\rho} < b,\; -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}.$$

5. Extension to Pseudoconvex Domains

We may now prove

Theorem 5.1. *Let Ω be an unbounded domain in $\mathbb{C}^n$ $(n \geq 3)$ with smooth boundary $b\Omega$ and M be a maximally complex closed $(2m+1)$-dimensional real submanifold $(m \geq 1)$ of $b\Omega$. Assume that*

(i) *$b\Omega$ is weakly pseudoconvex and the Levi form $\mathcal{L}(b\Omega)$ has at least $n - m$ positive eigenvalues at every point of M;*
(ii) *M satisfies condition $(\star)$.*

Then there exists a unique $(m+1)$-dimensional complex analytic subvariety W of Ω, such that $bW = M$. Moreover the singular locus of W is discrete and the closure of W in $\overline{\Omega} \setminus \mathsf{Sing}\, W$ is a smooth submanifold with boundary M.

Proof. Assume, for the moment, that condition $(\star)$ is replaced by the stronger condition

$\overline{\Omega}^{\infty} \cap \Sigma_0 = \emptyset$ where $\overline{\Omega}^{\infty}$ denotes the projective closure of Ω.

The only thing we have to show in order to conclude the proof (by using the methods of the previous section) is that, up to a holomorphic change of coordinates and a holomorphic embedding $V : \mathbb{C}^n \to \mathbb{C}^N$, we can choose a sequence of real hyperplanes $H_k \subset \mathbb{C}^N$, $k \in \mathbb{N}$, which are exhaustive in the following sense:

1. $H_k \cap V(S)$ is compact, for all $k \in \mathbb{N}$;
2. one of the two halfspaces in which H_k divides $\mathbb{C}^N$, say H_k^+, intersects $V(\Omega)$ in a relatively compact set;
3. $\cup_k (H_k^+ \cap V(\Omega)) = V(\Omega)$.

The arguments of Proposition 4.2, indeed — excluded the proof that the singularities are isolated — depend only on the fact that we can cut M by complex hyperplanes, obtaining compact maximally complex submanifolds. Once we have found $W' \subset V(\mathbb{C}^n)$ (W' is in fact contained in $V(\mathbb{C}^n)$ by analytic continuation, since it has to coincide with the strip in a neighborhood of M), we set $W = V^{-1}(W')$. Observe that the hypersurfaces $V^{-1}(H_k)$ are an exhaustive sequence for Ω; let Ω_k be correspondent sequence of relatively compact subsets. Since Ω is a domain of holomorphy, for each k we can choose a strongly pseudoconvex open subset $\Omega'_k \subset \Omega$ such that $b\Omega'_k \cap \Omega_k \subset I$, where I is the tubular neighborhood found in Corollary 3.1. So, in each Ω_k we can suppose that we deal with a strongly pseudoconvex open set, and thus the proof of the fact that the singularities are isolated is the same as in Proposition 4.2.

Following [10], we divide the proof in two steps.

Step 1. P linear. We consider $\overline{\Omega}^{\infty} \subset \mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}_{\infty}$, which is disjoint from $\Sigma_0 = \{P = 0\}$. So we can consider new coordinates of $\mathbb{CP}^n$ in such a way that Σ_0 is the $\mathbb{CP}^{n-1}$ at infinity. Now Ω is a relatively compact open set of $(\mathbb{C}^n)' = \mathbb{CP}^n \setminus \Sigma_0$, and $H_\infty = \mathbb{CP}^{n-1}_{\infty} \cap (\mathbb{C}^n)'$ is a complex hyperplane containing the boundary of S. Let $H^{\mathbb{R}}_{\infty} \supset H_\infty$ be a real hyperplane. The intersection between S and a translated of $H^{\mathbb{R}}_{\infty}$ is either empty or compact. For all $z \in \Omega$, there exist a real hyperplane $H^{\mathbb{R}}_{\infty} \not\ni z$, intersecting Ω, and a small translated H_{ε_z} such that $z \in H^+_{\varepsilon_z}$. Since $\Omega = \cup_z (H^+_{\varepsilon_z} \cap \Omega)$, and Ω is a countable union of compact sets, we may choose an exhaustive sequence H_k.

Step 2. P generic. We use the Veronese map v to embed $\mathbb{CP}^n$ in a suitable $\mathbb{CP}^N$ in such a way that $v(\Sigma_0) = L_0 \cap v(\mathbb{CP}^n)$, where L_0 is a linear subspace. The Veronese map v is defined as follows: let d be the degree of P, and let

$$N = \binom{n+d}{d} - 1.$$

Then v is defined by

$$v(z) = v[z_0 : \ldots : z_n] = [\ldots : w_I : \ldots]_{|I|=d},$$

where $w_I = z^I$. If $P = \sum_{|I|=d} \alpha_I z^I$, then $v(\Sigma_0) = L_0 \cap v(\mathbb{CP}^n)$, where

$$L_0 = \left\{ \sum_{|I|=d} \alpha_I w_I = 0 \right\}.$$

Again we can change the coordinates so that L_0 is the $\mathbb{CP}^{N-1}$ at infinity. We may now find the exhaustive sequence H_k as in Step 1.

This achieves the proof in the case when $\overline{\Omega}^{\infty} \cap \Sigma_0 = \emptyset$. The general case is now an easy consequence.

Indeed, since $\mathbb{CP}^n \setminus \Sigma_0$ is Stein, there is a strictly plurisubharmonic exhaustion function ψ. The sets

$$\Omega_c = \{\psi < c\}$$

are an exhaustive strongly pseudoconvex family for $\mathbb{CP}^n \setminus \Sigma_0$. Thus in view of $(\star)$ there exists $\overline{c}$ such that $\overline{M} \subset \Omega_{\overline{c}}$. $\Omega' \doteqdot \Omega \cap \Omega_{\overline{c}}$, up to a regularization of the boundary, is a strongly pseudoconvex open set verifying $(\star)$ in whose boundary lies M, and thus M can be extended thanks to what has already been proved. $\square$

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