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Hyperbolicity in unbounded convex domains

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Abstract. We provide several equivalent characterizations of Kobayashi hyperbolicity in unbounded convex domains in terms of peak and anti-peak functions at infinity, affine lines, Bergman metric and iteration theory.

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1 Introduction

Despite the fact that linear convexity is not an invariant property in complex analysis, bounded convex domains in $\mathbb{C}^N$ have been very much studied as prototypes for the general situation.

In particular, by Harris' theorem [6] (see also, [1], [9]) it is known that bounded convex domains are always Kobayashi complete hyperbolic (and thus by Royden's theorem, they are also taut and hyperbolic). Moreover, by Lempert's theorem [10], [11], the Kobayashi distance can be realized by means of extremal discs. These are the basic cornerstone for many useful results, especially in pluripotential theory and iteration theory.

On the other hand, not much is known about unbounded domains. Clearly, the geometry at infinity must play some important role. In this direction, Gaussier [5] gave some conditions in terms of existence of peak and anti-peak functions at infinity for an unbounded domain to be hyperbolic, taut or complete hyperbolic. Recently, Nikolov and Pflug [14] deeply studied conditions at infinity which guarantee hyperbolicity, up to a characterization of hyperbolicity in terms of the asymptotic behavior of the Lempert function.

In these notes we restrict ourselves to the case of unbounded convex domains, where, strange enough, many open questions in the previous directions seem to be still open. In particular an unbounded convex domain needs not to be hyperbolic, as the example of $\mathbb{C}^k$ shows. Some estimates on the Caratheodory and Bergman metrics in convex domains were obtained by Nikolov and Pflug in [12], [13]. The question is whether one can understand easily hyperbolicity of unbounded convex domains in terms of geometric or analytic properties. A result in this direction was obtained by Barth [3], who proved the equivalence of properties (1), (2) and (6) in the theorem below.

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The aim of the present paper is to show that actually for unbounded convex domains, hyperbolicity can be characterized in many different ways and can be easily inferred just looking at a single boundary point.

The dichotomy we discovered for unbounded convex domains is rather stringent: either the domain behaves like a bounded convex domain or it behaves like $\mathbb{C}^k$. In particular, this provides examples of unbounded domains which admit the Bergman metric and are complete with respect to it.

The main result of these notes is the following (notations and terminology are standard and will be recalled in the next section):

Theorem 1.1. *Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) convex domain. The following are equivalent:*

(1) *D is biholomorphic to a bounded domain;*
(2) *D is (Kobayashi) hyperbolic;*
(3) *D is taut;*
(4) *D is complete (Kobayashi) hyperbolic;*
(5) *D does not contain nonconstant entire curves;*
(6) *D does not contain complex affine lines;*
(7) *D has N linearly independent separating real hyperplanes;*
(8) *D has peak and antipeak functions (in the sense of Gaussier) at infinity;*
(9) *D admits the Bergman metric b_D:*
(10) *D is complete with respect to the Bergman metric b_D;*
(11) *for any $f : D \to D$ holomorphic such that the sequence of its iterates $\{f^{\circ k}\}$ is not compactly divergent there exists $z_0 \in D$ such that $f(z_0) = z_0$.*

The first implications of the theorem allow to obtain the following canonical complete hyperbolic decomposition for unbounded convex domains, which is used in the final part of the proof of the theorem itself.

Proposition 1.2. *Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) convex domain. Then there exist a unique k $(0 \leq k \leq N)$ and a unique complete hyperbolic convex domain $D' \subset \mathbb{C}^k$, such that, up to a linear change of coordinates, $D = D' \times \mathbb{C}^{N-k}$.*

By using such a canonical complete hyperbolic decomposition, one sees for instance that the "geometry at infinity" of an unbounded convex domain can be inferred from the geometry of any finite point of its boundary (see the last section for precise statements). For example, as an application of Corollary 4.3 and Theorem 1.1, existence of peak and anti-peak functions (in the sense of Gaussier) for an unbounded convex domain equals the absence of complex line in the "CR-part" of the boundary of the domain itself. This answers a question in Gaussier's paper (see [5, pag. 115]) about geometric conditions for the existence in convex domains of peak and anti-peak plurisubharmonic functions at infinity.

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2 Preliminary

A convex domain $D \subset \mathbb{C}^N$ is a domain such that for any couple $z_0, z_1 \in D$ the real segment joining z_0 and z_1 is contained in D. It is well known that for any point $p \in \partial D$ there exists (at least) one real separating hyperplane $H_p = \{z \in \mathbb{C}^n : \mathrm{Re}\, L(z) = a\}$, with L a complex linear functional and $a \in \mathbb{R}$ such that $p \in H_p$ and $D \cap H_p = \emptyset$. Such a hyperplane H_p is sometimes also called a tangent hyperplane to D at p. We say that k separating hyperplanes $H_j = \{\mathrm{Re}\, L_j(z) = a_j\}, j = 1, \ldots, k$, are linearly independent if $L_1, \ldots, L_k$ are linearly independent linear functionals.

Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disc. Let $D \subset \mathbb{C}^N$ be a domain. The Kobayashi pseudo-metric for the point $z \in D$ and vector $v \in \mathbb{C}^N$ is defined as

$$\kappa_D(z; v) := \inf\{\lambda > 0 | \exists \varphi : \mathbb{D} \xrightarrow{\text{hol}} D, \varphi(0) = z, \varphi'(0) = v/\lambda\}.$$

If $\kappa_D(z; v) > 0$ for all $v \neq 0$ then D is said to be (Kobayashi) hyperbolic. The pseudo-distance k_D obtained by integrating κ_D is called the Kobayashi pseudodistance. The domain D is (Kobayashi) complete hyperbolic if k_D is complete.

The Carathéodory pseudo-distance c_D is defined by

$$c_D(z, w) = \sup\{k_{\mathbb{D}}(f(z), f(w)) : f : D \to \mathbb{D} \text{ holomorphic}\}.$$

In general, $c_D \leq k_D$.

We refer the reader to the book of Kobayashi [9] for properties of Kobayashi and Carathéodory metrics and distances.

Another (pseudo)distance that can be introduced on the domain D is the Bergman (pseudo) distance (see, e.g., [9, Sect. 10, Ch. 4]). Let $\{e_j\}$ be a orthonormal complete basis of the space of square-integrable holomorphic functions on D. Then let

$$l_D(z, \overline{w}) := \sum_{j=0}^{\infty} e_j(z)\overline{e}_j(\overline{w}).$$

If $l_D(z, \overline{z}) > 0$ one can define a symmetric form $b_D := 2 \sum h_{jk} dz_j \otimes d\overline{z}_k$, with $h_{jk} = \frac{\partial^2 \log b_D(z,\overline{z})}{\partial z_j, \partial \overline{z}_k}$, which is a positive semi-definite Hermitian form, called the Bergman pseudo-metric of D. If b_D is positive definite everywhere, one says that D admits the Bergman metric b_D. For instance, $\mathbb{C}^k$, $k \geq 1$ does not support square-integrable holomorphic functions, therefore $l_{\mathbb{C}^k} \equiv 0$ and $\mathbb{C}^k$ does not admit the Bergman metric.

For the next result, see [9, Corollaries 4.10.19, 4.10.20]:

Proposition 2.1. *Let $D \subset \mathbb{C}^N$ be a domain.*

(1) *Assume $b_D(z, \overline{z}) > 0$ for all $z \in D$. If c_D is a distance, if it induces the topology of D and if the c_D-balls are compact, then D admits the Bergman metric b_D and it is complete with respect to b_D.*

(2) *If D is a bounded convex domain then it (admits the Bergman metric and it) is complete with respect to the Bergman metric.*

Let G be another domain. We recall that if $\{\varphi_k\}$ is a sequence of holomorphic mappings from G to D, then the sequence is said to be compactly divergent if for any two compact sets $K_1 \subset G$ and $K_2 \subset D$ it follows that $\sharp\{k \in \mathbb{N} : \varphi_k(K_1) \cap K_2 \neq \emptyset\} < +\infty$.

A family F of holomorphic mappings from G to D is said to be normal if each sequence of F admits a subsequence which is either compactly divergent or uniformly convergent on compacta.

If the family of all holomorphic mappings from the unit disc $\mathbb{D}$ to D is normal, then D is said to be taut. It is known:

Theorem 2.2. *Let $D \subset \mathbb{C}^N$ be a domain.*
(1) *(Royden) D complete hyperbolic $\Rightarrow$ D taut $\Rightarrow$ D hyperbolic.*
(2) *(Kiernan) If D is bounded then D is hyperbolic.*
(3) *(Harris) If D is a bounded convex domain then D is complete hyperbolic.*

The notion of (complete) hyperbolicity is pretty much related to existence of peak functions at each boundary point. In case D is an unbounded domain, H. Gaussier [5] introduced the following concepts of "peak and antipeak functions" at infinity, which we use in the sequel:

Definition 2.3. A function $\varphi : \overline{D} \to \mathbb{R} \cup \{-\infty\}$ is called a *global peak plurisubharmonic function at infinity* if it is plurisubharmonic on D, continuous up to $\overline{D}$ (closure in $\mathbb{C}^N$) and

$$\begin{cases} \lim_{z\to\infty} \varphi(z) = 0, \\ \varphi(z) < 0 \quad \forall z \in \overline{D}. \end{cases}$$

A function $\varphi : \overline{D} \to \mathbb{R} \cup \{-\infty\}$ is called a *global antipeak plurisubharmonic function at infinity* if it is plurisubharmonic on D, continuous up to $\overline{D}$ and

$$\begin{cases} \lim_{z\to\infty} \varphi(z) = -\infty, \\ \varphi(z) > -\infty \quad \forall z \in \overline{D}. \end{cases}$$

For short we will simply call them peak and antipeak functions (in the sense of Gaussier) at infinity.

Gaussier proved the following result:

Theorem 2.4 (Gaussier)**.** *Let $D \subset \mathbb{C}^N$ be an unbounded domain. Assume that D is locally taut at each point of ∂D and there exist peak and antipeak functions (in the sense of Gaussier) at infinity. Then D is taut.*

Obviously a convex domain is locally taut at each boundary point, thus tautness follows from existence of peak and antipeak functions (in the sense of Gaussier) at infinity.

Finally, if $f : D \to D$ is a holomorphic function, the sequence of its iterates $\{f^{\circ k}\}$ is defined by induction as $f^{\circ k} := f^{\circ(k-1)} \circ f$. If f has a fixed point $z_0 \in D$, then $\{f^{\circ k}\}$ is not compactly divergent. On the other hand, depending on the geometry of D, there exist examples of holomorphic maps f such that $\{f^{\circ k}\}$ is not compactly divergent but f has no fixed points in D. It is known (see [2]) that

Theorem 2.5 (Abate). *Let $D \subset \mathbb{C}^N$ be a taut domain. Assume that $H^j(D;\mathbb{Q}) = 0$ for all $j > 0$ and let $f : X \to X$ holomorphic. Then the sequence of iterates $\{f^{\circ k}\}$ is compactly divergent if and only if f has no periodic points in D.*

If D is a bounded convex domain then the sequence of iterates $\{f^{\circ k}\}$ is compactly divergent if and only if f has no fixed points in D.

3 The proof of Theorem 1.1

The proof of Theorem 1.1 is obtained in several steps, which might be of some interest by their own.

For a domain $D \subset \mathbb{C}^N$ let us denote by δ_D the Lempert function given by

$$\delta_D(z,w) = \inf\{\omega(0,t) : t \in (0,1), \exists \varphi \in \mathsf{Hol}(\mathbb{D}, D) : \varphi(0) = z, \varphi(t) = w\}.$$

The Lempert function is not a pseudodistance in general because it does not enjoy the triangle inequality. The Kobayashi pseudodistance is the largest minorant of δ_D which satisfies the triangle inequality. The following lemma (known as Lempert's theorem in case of bounded convex domains) is probably known, but we provide its simple proof due to the lack of reference.

Lemma 3.1. *Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) convex domain. Then $k_D = \delta_D = c_D$.*

Proof. The result is due to Lempert [11] in case D is bounded. Assume D is unbounded. Let D_R be the intersection of D with a ball of center the origin and radius $R > 0$. For $R >> 1$ the set D_R is a nonempty convex bounded domain. Therefore $k_{D_R} = \delta_{D_R} = c_{D_R}$. Now $\{D_R\}$ is an increasing sequence of domains whose union is D. Hence, $\lim_{R\to\infty} k_{D_R} = k_D$, $\lim_{R\to\infty} c_{D_R} = c_D$ and $\lim_{R\to\infty} \delta_{D_R} = \delta_D$ (see, e.g., [7, Prop. 2.5.1] and [7, Prop. 3.3.5]). Thus $k_D = \delta_D = c_D$. □

Proposition 3.2. *Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) convex domain. Then the Kobayashi balls in D are convex.*

Proof. For the bounded case, see [1, Proposition 2.3.46]. For the unbounded case, let B_ε be the Kobayashi ball of radius ε and center $z_0 \in D$, let D_R be the intersection of D with an Euclidean ball of center the origin and radius $R > 0$, and let B_ε^R be the Kobayashi ball in D_R of radius ε and center z_0. Then the convex sets $B_\varepsilon^R \subset B_\varepsilon^{R+\delta} \subset B_\varepsilon$ for all $R >> 1$, $\delta > 0$, and their convex increasing union $\cup_R B_\varepsilon^R = B_\varepsilon$, since $\lim_{R\to\infty} k_{D_R} = k_D$. □

Lemma 3.3. *Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) taut convex domain. Then for any couple $z, w \in D$ there exists $\varphi \in \mathsf{Hol}(\mathbb{D}, D)$ such that $\varphi(0) = z, \varphi(t) = w$ $t \in [0,1)$ and $k_D(z,w) = \omega(0,t)$.*

Proof. By Lemma 3.1, $k_D = \delta_D$, so there exists a sequence $\{\varphi_k\}$ of holomorphic discs and $t_k \in (0,1)$ such that $\varphi_k(0) = z$ and $\varphi_k(t_k) = w$ and

$$k_D(z,w) = \lim_{k\to\infty} \omega(0,t_k).$$

Since D is taut and $\varphi_k(0) = z$ for all k, we can assume that $\{\varphi_k\}$ converges uniformly on compacta to a (holomorphic) map $\varphi : \mathbb{D} \to D$. Then $\varphi(0) = z$. Moreover, since $k_D(z,w) < \infty$, there exists $t_0 < 1$ such that $t_k \leq t_0$ for all k. We can assume (up to subsequences) that $t_k \to t \leq t_0$. Then

$$k_D(z,w) = \lim_{k\to\infty} \omega(0,t_k) = \omega(0,t).$$

Moreover, $\varphi(t) = \lim_{k\to\infty} \varphi_k(t_k) = w$ and we are done. □

Proposition 3.4. *Let $D \subset \mathbb{C}^N$ be a (possibly unbounded) convex domain. Then D is taut if and only if it is complete hyperbolic.*

Proof. One direction is contained in Royden's theorem. Conversely, assume that D is taut. We are going to prove that every closed Kobayashi balls is compact (which is equivalent to be complete hyperbolic, see [9] or [1, Proposition 2.3.17]).

Let $R > 0$, $z \in D$ and let $B(z,R) = \{w \in D : k_D(z,w) \leq R\}$. If $B(z,R)$ is not compact then there exists a sequence $\{w_k\}$ such that $w_k \to p \in \partial D \cup \{\infty\}$ and $k_D(z,w_k) \leq R$. For any k, let $\varphi_k \in \mathsf{Hol}(\mathbb{D}, D)$ be the extremal disc given by Lemma 3.3 such that $\varphi_k(0) = z$, $\varphi_k(t_k) = w_k$ for some $t_k \in (0,1)$ and $k_D(z,w_k) = \omega(0,t_k)$.

Notice that, since $k_D(z,w_k) \leq R$, then there exists $t_0 < 1$ such that $t_k \leq t_0$ for all k. We can assume up to subsequences that $t_k \to t$ with $t < 1$. Since D is taut and $\varphi_k(0) = z$, up to extracting subsequences, the sequence $\{\varphi_k\}$ is converging uniformly on compacta to a holomorphic disc $\varphi : \mathbb{D} \to D$ such that $\varphi(0) = z$. However,

$$\varphi(t) = \lim_{k\to\infty} \varphi_k(t_k) = \lim_{k\to\infty} w_k = p,$$

a contradiction. Therefore $B(z,R)$ is compact and D is complete hyperbolic. □

For the next proposition, cfr. [4, Lemma 3].

Proposition 3.5. *Let $D \subset \mathbb{C}^N$ be a convex domain, which does not contain complex affine lines. Then there exist $\{L_1 = 0\}, \ldots, \{L_N = 0\}$ linearly independent complex hyperplanes containing the origin and $a_1, \ldots, a_N \in \mathbb{R}$ such that*

$$D \subset \{\mathsf{Re}\, L_1 > a_1, \ldots, \mathsf{Re}\, L_N > a_N\}.$$

Proof. Without loss of generality we can assume that $O \in D$. Since D does not contain complex affine lines, ∂D is not empty. Take a point $p_1 \in \partial D$ and a tangent real hyperplane through p_1, given by $\{\mathsf{Re}\, L_1 = a_1\}$ (if the boundary is smooth there is only one tangent hyperplane), where L_1 is defined so that $D \subset \{\mathsf{Re}\, L_1 > a_1\}$.

Suppose that $L_1, \ldots, L_k$, $k < N$, are already defined, they are linearly independent and

$$D \subset \{\mathsf{Re}\, L_1 > a_1, \ldots, \mathsf{Re}\, L_k > a_k\}.$$

The intersection $h_k = \cap_1^k \{L_i = 0\}$ is a complex $(N-k)$-dimensional plane through the origin O (which is also contained in D by hypothesis). Since D does not contain complex affine lines, $\partial D \cap h_k$ is not empty. Take a point $p_{k+1} \in \partial D \cap h_k$ and consider a tangent real hyperplane through p_{k+1}, $\{\mathrm{Re}\, L_{k+1} = a_{k+1}\}$, where L_{k+1} is defined so that $D \subset \{\mathrm{Re}\, L_{k+1} > a_{k+1}\}$. By construction L_{k+1} is linearly independent from $L_1, \ldots, L_k$ and

$$D \subset \{\mathrm{Re}\, L_1 > a_1, \ldots, \mathrm{Re}\, L_{k+1} > a_{k+1}\}.$$

Continuing this way, the proof is concluded. □

Now we are in a good shape to prove part of Theorem 1.1:

Proposition 3.6. *Let $D \subset \mathbb{C}^N$ be a convex domain. The following are equivalent:*
(1) *D is biholomorphic to a bounded domain;*
(2) *D is (Kobayashi) hyperbolic;*
(3) *D is taut;*
(4) *D is complete (Kobayashi) hyperbolic;*
(5) *D does not contain nonconstant entire curves;*
(6) *D does not contain complex affine lines;*
(7) *D has N linearly independent separating real hyperplanes;*
(8) *D has peak and antipeak functions (in the sense of Gaussier) at infinity;*

Proof.
(1) ⇒ (2): every bounded domain in $\mathbb{C}^N$ is hyperbolic by [8] (see, also, [1, Thm. 2.3.14])

(2) ⇒ (5) ⇒ (6): obvious.

(6) ⇒ (7): it is Proposition 3.5.

(7) ⇒ (1): let $L_1, \ldots, L_N$ be linearly independent complex linear functionals and let $a_1, \ldots, a_N \in \mathbb{R}$ be such that $\{\mathrm{Re} L_j = a_j\}$ for $j = 1, \ldots, N$ are real separating hyperplanes for D. Up to sign changes, we can assume that $D \subset \{\mathrm{Re} L_j > a_j\}$. Then the map

$$F(z_1, \ldots, z_N) := \left(\frac{1}{L_1(z) - a_1 + 1}, \ldots, \frac{1}{L_N(z) - a_N + 1}\right)$$

maps D biholomorphically on a bounded convex domain of $\mathbb{C}^N$.

(6) ⇒ (8): let $L_1, \ldots, L_N$ be as in Proposition 3.5. Up to a linear change of coordinates, we can suppose that $z_j = L_j$ for all $1 \le j \le N$. A peak function is given by

$$-\mathrm{Re} \sum_{j=1}^{N} \frac{1}{z_j - a_j + 1}.$$

Let $D_j := \{\mathrm{Re} L_j > a_j\}$. Then $D \subset \prod_{j=1}^{N} D_j$, and D_j is biholomorphic to $\mathbb{D}$ for each j. In particular $\mathbb{C} \setminus D_j$ is not a polar set. We may assume that $0 \notin D_j$. Let G_j be the image of D_j under the transformation $z \to 1/z$. Since $\mathbb{C} \setminus G_j$ is not a polar set, there exists $\varepsilon > 0$ such that $\mathbb{C} \setminus G_j^{\varepsilon}$ is not polar, too, where $G_j^{\varepsilon} = G_j \cup \varepsilon\mathbb{D}$. Denote by g_j^{ε} the Green function of G_j^{ε}. Then $h_j = g_j^{\varepsilon}(0;\cdot)$ is a negative harmonic function on G_j with $\lim_{z \to 0} h_j(z) = -\infty$ and $\inf_{G_j \setminus r\mathbb{D}} h_j > -\infty$ for any $r > 0$. Then $\psi_j(z) = h_j(1/z)$ is an antipeak function of D_j at ∞ and hence $\psi = \sum_{j=1}^{N'} \psi_j$ is an antipeak function for D at ∞.

(8) ⇒ (3): it is Gaussier's theorem [5, Prop. 2].

(3) $\Rightarrow$ (4): it is Proposition 3.4.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2): it is Royden's theorem [15, Prop. 5, pag. 135 and Corollary p.136]. $\square$

As a consequence we have Proposition 1.2, which gives a *canonical complete hyperbolic decomposition* of a convex domain as the product of a complete hyperbolic domain and a copy of $\mathbb{C}^k$.

Proof of Proposition 1.2. We prove the result by induction on N. If $N = 1$ then either $D = \mathbb{C}$ or D is biholomorphic to the disc and hence (complete) hyperbolic.

Assume the result is true for N, we prove it holds for $N+1$. Let $D \subset \mathbb{C}^{N+1}$ be a convex domain. Then, by Proposition 3.6, either D is complete hyperbolic or D contains an affine line, say, up to a linear change of coordinates

$$l_{N+1} = \{z_1, \ldots, z_N = 0\} \subset D.$$

Clearly, there exists $c \in \mathbb{C}$ such that $D \cap \{z_{N+1} = c\} \neq \emptyset$. Up to translation we can assume $c = 0$. Let us define

$$D_N = D \cap \{z_{N+1} = 0\}.$$

$D_N \subset \mathbb{C}^N$ is convex. We claim that $D = D_N \times \mathbb{C}$. Induction will then conclude the proof.

Let $z_0 \in D_N$. We want to show that $(z_0, \zeta) \in D$ for all $\zeta \in \mathbb{C}$. Since $l_{N+1} \subset D$ then $(0, \zeta) \in D$ for all $\zeta \in \mathbb{C}$. Assume $z_0 \neq 0$. Fix $\zeta \in \mathbb{C}$. Since D_N is open, there exists $\varepsilon_0 > 0$ such that $z_1 := (1+\varepsilon_0)z_0 \in D_N$. Since D is convex, for any $t \in [0,1]$ it follows $t(z_1, 0) + (1-t)(0, \xi) \in D$ for all $\xi \in \mathbb{C}$. Setting $\xi_0 := \frac{1+\varepsilon_0}{\varepsilon_0}\zeta \in \mathbb{C}$ and $t_0 = (1+\varepsilon_0)^{-1} \in (0,1)$ we obtain

$$(z_0, \zeta) = t_0\,(z_1, 0) \;+\; (1-t_0)\,(0, \xi_0) \in D,$$

completing the proof. $\square$

In order to finish the proof of Theorem 1.1 we need to show that the first eight conditions, which are all and the same thanks to Proposition 3.6, are equivalent to (9), (10), (11).

Proof of Theorem 1.1. Conditions (1) to (8) are all equivalent by Proposition 3.6.

(10)$\Rightarrow$(9): obvious.

(4)$\Rightarrow$(10): By Lemma 3.1, the Caratheodory distance c_D equals the Kobayashi distance k_D, thus, since D is (Kobayashi) complete hyperbolic, c_D is a distance which induces the topology on D and the c_D-balls are compact. By Proposition 2.1 then D admits the Bergman metric and it is complete with respect to it.

(9)$\Rightarrow$(4): Assume D is not complete hyperbolic. Then by Proposition 1.2, up to a linear change of coordinates, $D = D' \times \mathbb{C}^k$ for some complete hyperbolic domain D' and $k \geq 1$. By the product formula (see [9, Prop. 4.10.17]) $l_D = l_{D'} \cdot l_{\mathbb{C}^k} \equiv 0$ and thus D does not admit the Bergman metric.

(11) $\Rightarrow$ (4): Assume D is not complete hyperbolic. We have to exhibit a holomorphic self-map $f : D \to D$ such that $\{f^{\circ k}\}$ is not compactly divergent but there exists no $z_0 \in D$ such that $f(z_0) = z_0$.

By Proposition 1.2, up to a linear change of coordinates, $D = D' \times \mathbb{C}^k$ for some complete hyperbolic domain D' and $k \geq 1$. Let

$$f : D' \times \mathbb{C}^{k-1} \times \mathbb{C} \ni (z, w', w) \mapsto (z, w', e^w + w) \in D' \times \mathbb{C}^{k-1} \times \mathbb{C}.$$

Then clearly f has no fixed points in D. However, if $w_0 = \log(i\pi)$, then $f^{\circ 2}(z, w', w_0) = (z, w', w_0)$, and therefore the sequence $\{f^{\circ k}\}$ is not compactly divergent.

(4) $\Rightarrow$ (11): According to the theory developed so far, if D is complete hyperbolic, then it is taut and its Kobayashi balls are convex and compact. With these ingredients, the proof for bounded convex domains go through also in the unbounded case (see [1, Thm. 2.4.20]). □

4 Applications

Corollary 4.1. *Let $D \subset \mathbb{C}^N$ be a convex domain. If there exists a point $p \in \partial D$ such that ∂D is strongly convex at p then D is complete hyperbolic.*

Proof. By Proposition 1.2, if D were not complete hyperbolic, up to linear changes of coordinates, $D = D' \times \mathbb{C}^k$ for some complete hyperbolic convex domain D' and $k \geq 1$. Then $\partial D = \partial D' \times \mathbb{C}^k$ could not be strongly convex anywhere. □

Note that the converse to the previous corollary is false: the half-plane $\{\zeta \in \mathbb{C} : \mathsf{Re}\,\zeta > 0\}$ is a complete hyperbolic convex domain in $\mathbb{C}$ with boundary which is nowhere strongly convex.

Proposition 4.2. *Let $D \subset \mathbb{C}^N$ be an (unbounded) convex domain. Then D has canonical complete hyperbolic decomposition (up to a linear change of coordinates) $D = D' \times \mathbb{C}^k$ with D' complete hyperbolic, if and only if for every $p \in \partial D$ and every separating hyperplane H_p, $(H_p \cap \partial D) \cap i(H_p \cap \partial D)$ contains a copy of $\mathbb{C}^k$ but contains no copies of $\mathbb{C}^{k+1}$.*

Proof. ($\Rightarrow$) Since $D = D' \times \mathbb{C}^k$, for every $p \in \partial D$ and every separating hyperplane H_p,

$$(H_p \cap \partial D) \cap i(H_p \cap \partial D) = [(H_p \cap \partial D') \cap i(H_p \cap \partial D')] \times \mathbb{C}^k.$$

Since D' is complete hyperbolic, its boundary does not contain complex lines.

($\Leftarrow$) Since D is convex, $D = D' \times \mathbb{C}^{k'}$, by Proposition 1.2. By the first part of the present proof, for every $p \in \partial D$ and every separating hyperplane H_p,

$$(H_p \cap \partial D) \cap i(H_p \cap \partial D) = \mathbb{C}^{k'}.$$

Hence $k' = k$. □

Corollary 4.3. *Let $D \subset \mathbb{C}^N$ be an (unbounded) convex domain. If there exist $p \in \partial D$ and a separating hyperplane H_p such that $(H_p \cap \partial D) \cap i(H_p \cap \partial D)$ does not contain any complex affine line then D is complete hyperbolic. Conversely, if D is complete hyperbolic, then for any point $p \in \partial D$ and any separating hyperplane H_p, it follows that $(H_p \cap \partial D) \cap i(H_p \cap \partial D)$ does not contain any complex affine line.*

As a final remark, we notice that, as Abate's theorem 2.5 is the cornerstone to the study of iteration theory in bounded convex domains, our Theorem 1.1 and Proposition 1.2 can be used effectively well to the same aim for unbounded convex domains. In fact, if $D = D' \times \mathbb{C}^k$ is the canonical complete hyperbolic decomposition of D, then a holomorphic self map $f : D \to D$ can be written in the coordinates $(z, w) \in D' \times \mathbb{C}^k$ as $f(z, w) = (\varphi(z, w), \psi(z, w))$, where $\varphi : D' \times \mathbb{C}^k \to D'$ and $\psi : D' \times \mathbb{C}^k \to \mathbb{C}^k$. In particular, since D' is complete hyperbolic, then φ depends only on z, namely, $f(z, w) = (\varphi(z), \psi(z, w))$. The map $\psi(z, w)$ can be as bad as entire functions in $\mathbb{C}^k$ are, but the map φ is a holomorphic self-map of a complete hyperbolic convex domains and its dynamics goes similarly to that of holomorphic self-maps of bounded convex domains. For instance, if the sequence $\{f^{\circ k}\}$ is non-compactly divergent, then f might have no fixed points, but the sequence $\{\varphi^{\circ k}\}$ must have at least one.

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