CHAPTER 1

The interplay between topological algebra theory and algebras of holomorphic functions.

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Abstract There are several interactions between the general theory of commutative topological algebras over $\mathbb{C}$ with unity and that of the holomorphic functions of one or several complex variables.
This chapter is intended to be a review chapter of classical results in this subjects and a list of very difficult and long-standing open problems. A big reference section will provide the reader material to dwell into the proofs of theorems and understand more deeply the subjects.
This chapter will be divided in two parts.
In the first we will review known results on the general theory of Banach algebras, Frechet algebras, $\mathcal{LB}$ and $\mathcal{LF}$ algebras, as well as the open problems in the field.
In the second part we will examine *concrete* algebras of holomorphic functions, again reviewing known results and open problems.
We will mainly focus on the interplay between the two theories, showing how each one can be considered either an object of study per se or a mean to understand better the other subject.

1.1 Topological algebras

A *topological algebra* A over a topological field $\mathbb{K}$ is a topological space endowed with continuous operations

$$+ : A \times A \to A\,,$$

$$\cdot : \mathbb{K} \times A \to A\,,$$

$$\star : A \times A \to A\,,$$

such that $(A, +, \cdot)$ is a vector space over $\mathbb{K}$ and $(A, +, \star)$ is an algebra. If the inner product $\star$ is commutative, we will call A a *commutative (topological) algebra*. We will usually omit the adjective topological in the sequel. A is said commutative algebra with unit if there is $\mathbf{1} \in A$ such that $\mathbf{1} \star a = a \star \mathbf{1} = a$ for each $a \in A$.

Topological algebras were introduced in 1931 by David van Dantzig in his doctoral theses [47] and they were extensively studied by Izrail Gelfand, Mark Naĭmark and Georgi Šilov starting from the Forties.

Since we are interested in algebras of holomorphic functions, usually $\mathbb{K} = \mathbb{C}$, with the Euclidean topology. Moreover we will usually denote $a \star b$ simply by ab.

1.1.1 Banach algebras

A commutative algebra B over $\mathbb{C}$ with unit $\mathbf{1}$ is said to be a *Banach algebra* if it is a Banach space with a norm $\| \cdot \|$ such that

$$\| \mathbf{1} \| = 1, \quad \| xy \| \leq \| x \| \| y \|,$$

for all $x, y \in B$.

Algebras of continuous $\mathbb{C}$-valued bounded functions on a set D, containing constant functions, endowed with the supremum norm, are Banach algebras.

Banach algebras of $\mathbb{C}$-valued functions are such that for all elements b of the algebra

$$\| b^2 \| = \| b \|^2 \ . \tag{1.1}$$

A Banach algebra B satisfying the above condition is called an *uniform algebra*. Actually this condition is very restrictive and the only uniform algebras are algebras of $\mathbb{C}$-valued functions.

From now on, B will always denote a Banach algebra.

The *spectrum* of an element $b \in B$ is the subset of $\mathbb{C}$

$$\sigma(b) \ = \ \{\lambda \in \mathbb{C} \ : \ \lambda \cdot \mathbf{1} - \ \text{is not invertible}\}$$

For every $b \in B$, the spectrum $\sigma(b)$ is a non empty compact subset of $\mathbb{C}$. This implies that the only Banach field (up to an isometric isomorphism) is $\mathbb{C}$ (Gelfand-Mazur theorem).

1.1.1.1 Banach algebras: characters and spectrum

A *character* of B is a homomorphism of algebras $\chi : B \to \mathbb{C}$. Every character χ of a Banach algebra is continuous. In particular the norm of every character is bounded by 1. If the character χ is not trivial (i.e. does not send B in 0) than since $\chi(\mathbf{1}) = \mathbf{1}$ $\| \chi \| = 1$.

The *spectrum* of B is the set $\mathfrak{M}(B)$ of all non trivial characters of B. If B^* denotes the dual space of B than

$$\mathfrak{M}(B) \ \subset \ \{b^* \in B^* \ : \| b^* \| \leq 1\},$$

thus the spectrum, with the topology induced by the weak-* topology on B^* (called the *Gelfand topology*), is compact, thanks to Banach-Alaoglu.

$\mathfrak{M}(B)$ equipped with the Gelfand topology is a compact Hausdorff space.

For every $b \in B$, the map $\hat{b} : \mathfrak{M}(B) \to \mathbb{C}$ defined by

$$\hat{b}(\chi) \;=\; \chi(b)$$

is called the *Gelfand transform* of b. By $\hat{B}$ we denote the set of all Gelfand transforms of B.

The Gelfand topology is the weakest topology on the spectrum $\mathfrak{M}(B)$ that makes every Gelfand transform a continuous map.

The map $\Gamma : B \to \mathcal{C}^0(\mathfrak{M}(B))$ sending an element of B to its Gelfand transform is continuous (the space of continuous $\mathbb{C}$-valued functions on the spectrum being endowed with the sup norm). If B is a uniform algebra, see (1.1), then this map makes $\hat{B}$ a function algebra on $\mathfrak{M}(B)$ isomorphic to B.

1.1.1.2 Banach algebras: maximal spectrum

The closure of any proper ideal of a Banach algebra B is a proper ideal. Hence maximal ideals are closed. The set of all maximal ideals of B, $\Omega(B)$, is called the *maximal spectrum* of B.

There is a natural bijection between the spectrum and the maximal spectrum of an algebra, sending a non trivial character to its kernel:

$$T: \;\mathfrak{M}(B) \;\to\; \Omega(B), \;\varphi \;\mapsto\; \operatorname{Ker}\varphi\,.$$

From now on, by $\mathfrak{M}(B)$ we will denote both the spectrum and the maximal spectrum of B, via the above identification.

The spectrum of a Banach algebra is a non-empty compact space.

1.1.1.3 Banach algebras: boundaries

A closed subset E of the spectrum $\mathfrak{M}(B)$ is called a *boundary* for B if for every $b \in B$, its Gelfand trasform $\hat{b}$ attains its maximum on E:

$$\max_E |\hat{b}| \;=\; \max_{\mathfrak{M}(B)} |\hat{b}|\,.$$

$\mathfrak{M}(B)$ is obviously a boundary for B.

If B is an algebra of functions on a compact K, then we can see each point $x \in K$ as a point of the spectrum (as the character of evaluation at x: $\varphi_x : b \to b(x)$), and obviously K is a boundary for B.

The intersection of all boundaries of B is itself a boundary, called the *Šilov boundary* of B. We will denote by γB the Šilov boundary of B.

1.1.1.4 Banach algebras: analytical properties of the spectrum

Many properties of the spectrum of a Banach algebra resemble basic properties of a domain D of $\mathbb{C}^n$, where the holomorphic functions on the domain D are the equivalent of the elements of the Banach algebra. These properties are thus usually named *analytic properties of the spectrum.*

Let B be a Banach algebra and $\mathfrak{M}(B)$ its spectrum. A character $\varphi \in \mathfrak{M}(B)$ is called a *peak point* for the Banach algebra B if there is an element $b \in B$ such that $\varphi(b) = 1$ and $|\psi(b)| < 1$ for all $\psi \in \mathfrak{M}(B)$, $\psi \neq \varphi$. Notice that the set of peak points for B is a subset of the Šilov boundary of B, since given a peak point $\varphi \in \mathfrak{M}(B)$, $\hat{b}$ attains its maximum only in φ.

A character $\varphi \in \mathfrak{M}(B)$ is called a *local peak point* for the Banach algebra B if there is a neighbourhood $U \subset \mathfrak{M}(B)$ of φ and an element $b \in B$ such that $\varphi(b) = 1$ and $|\psi(b)| < 1$ for all $\psi \in U$, $\psi \neq \varphi$.

If A is a uniform Banach algebra (i.e. an algebra of functions) then for any $U \subset \mathfrak{M}(B)$ not touching the Šilov boundary of A $U \cap \gamma A = \emptyset$ we define the algebra $A(U)$ as the closure of $A|_{\overline{U}}$ in the sup norm on $\overline{U}$. Hugo Rossi in 1959 [37] proved the following

Theorem 1.1 Let A be a uniform Banach algebra, and $U \subset \mathfrak{M}(A) \setminus \gamma A$. Any point $x \in U$ is not a local peak point for $A(\overline{U})$.

This theorem, called *maximum modulus principle* asserts that for uniform algebras points out of the Šilov boundary are not even local peak points. This closely resembles the maximum modulus principle for holomorphic functions (non-constant holomorphic functions have no local maxima).

Rossi's maximum modulus principle strongly suggests something holomorphic is going on. Thus many attempts to put on (a subdomain of) the spectrum of a uniform Banach algebra a complex structure making the elements of the algebra holomorphic functions have been made. The maximum modulus principle made reasonable to suppose that the spectrum of a uniform Banach algebra, except for the Šilov boundary, must in some sense have an underlying analytic structure.

Analytic structure of the spectrum: a negative result. In 1963 Stolzenberg, in a paper with the evocative title *An hull without anaytic structure* [45], gave a negative result.

Let $K \subset \mathbb{C}^n$ be a compact set and $\mathcal{Q}(K)$ the closure of polynomials in $\mathcal{C}^0(K)$. Stolzenberg considers a relatively weak notion of analytic structure: given a point $z \in \mathfrak{M}(\mathcal{Q}(K))$, (S_z, f_z) is said to be an *analytic structure through* z if

1. there is $N \in \mathbb{Z}^+$ such that $S_z \subset \mathbb{C}^N$ is a connected analytic subset containing the origin;
2. $f_z : S_z \to \mathbb{C}^n$ is a non-constant holomorphic map such that $f_z(0) = z$ and $f_z(S_z) \subset \mathfrak{M}(\mathcal{Q}(K))$.

Any subset of the spectrum $\mathfrak{M}(\mathcal{Q}(K))$ admitting such an analytic structure satisfies the maximum modulus principle.

Then Stolzenberg constructs a set so bad that nowhere satisfies such a definition and uses it to construct a counterexemple: let $P = \overline{\Delta} \times \overline{\Delta}$ be the closed bidisc of $\mathbb{C}^2$ and $\{z_j\}_{j\in\mathbb{N}} \subset \Delta \setminus \{0\}$ a countable dense sequence of points. Depending on z_j, Stolzenberg constructs a sequence of varieties in the bidisc P. The varieties V_j, as j goes to infinity, are more and more wavy so that their limit V is so bad that nowhere admits an analytic structure. Considering $K = V \cap bP$, one has $\mathfrak{M}(\mathcal{Q}(K)) \supset V$ and no subset of $\mathfrak{M}(\mathcal{Q}(K))$ admits an analytic structure.

Analytic structure of the spectrum: positive results. One year later, in 1964, Andrew Gleason [24] proved a positive results for points of the spectrum corresponding to finitely generated ideals of the algebra B.

More precisely, let us as usual denote both the spectrum and the maximal spectrum by $\mathfrak{M}(B)$, so that $m \in \mathfrak{M}(B)$ will mean both a maximal ideal of B and a character on B.

Theorem 1.2 (Gleason, 1964) Let B be a Banach algebra and $m_0 \in \mathfrak{M}(B)$ be a maximal ideal of B finitely generated by $b_1, \ldots, b_n \in B$. Then we can define the map $\varphi : \mathfrak{M}(B) \to \mathbb{C}^n$ by

$$\varphi(m) = (\hat{b}_1(m), \cdots, \hat{b}_n(m)) .$$

In this way $\varphi(m_0) = 0$. There is a neighbourhood of $0 \in \mathbb{C}^n$ such that

1. $\varphi|_{\varphi^{-1}(U)}$ is a homeomorphism onto a closed analytic subset of U;
2. the Gelfand transforms of elements of B are holomorphic functions on $\varphi^{-1}(U)$.

Moreover any maximal ideal $m \in \varphi^{-1}(U)$ is generated by $a_1(m), \ldots, a_n(m)$, where $a_j(m) = a_j - \hat{a}_j(m) \cdot \mathbf{1}$.

The key point in the proof of Gleason's theorem is a holomorphic version of the implicit function theorem which holds for Banach spaces.

By Gleason's theorem, if all maximal ideals of B are finitely generated, then the spectrum $\mathfrak{M}(B)$ is a compact complex space on which the holomorphic functions in $\hat{B}$ separate points. Thus $\mathfrak{M}(B)$ must be a finite set and B is a semi-local ring.

Finiteness properties of a topological algebra and structure properties of its spectrum have interesting links. Some of these were investigated by Artur Vaz Ferreira and Giuseppe Tomassini in [20].

Gleason's theorem was later generalized by Andrew Browder in 1971 [9]:

Theorem 1.3 (Browder, 1971) Let B be a Banach algebra and $m \in \mathfrak{M}(B)$ a maximal ideal of B. If

$$\dim_{\mathbb{C}} m/m^2 = n < +\infty , \tag{1.2}$$

then there is a neighbourhood $U \ni m$ in $\mathfrak{M}(B)$, an analytic subset V of the unit polydisc of $\mathbb{C}^n$ and a surjective homeomorphism $\tau : U \to V$ such that $\hat{b} \circ \tau$ is holomorphic on V, for all $b \in B$.

Clearly, if m is finitely generated as in the hypothesis of Gleason's theorem, then the hypothesis of Browder's theorem are fulfilled and near m the spectrum is analytic. Gleason's theorem has thus a stronger hypothesis, but gives something more, i.e. the fact that near m also the other maximal ideals are finitely generated, while Browder's theorem does not state nothing similar.

The finiteness hypothesis in Browder's theorem can be restated in terms of derivations. As usual, let B be a Banach algebra and $\varphi \in \mathfrak{M}(B)$ a point of its spectrum. A *derivation* of B at φ is a linear functional $\delta : B \to \mathbb{C}$ satisfying the Leibnitz rule, i.e. for all $a, b \in B$

$$\delta(ab) \,=\, \delta(a)\hat{b}(\varphi) + \hat{a}(\varphi)\delta(b)\,.$$

Considering $m = \operatorname{Ker}\varphi$ (i.e. m is the maximal ideal of B corresponding to φ), we have that a linear $\delta : B \to \mathbb{C}$ is a derivation if and only if $\operatorname{Ker}\delta$ contains the unit $\mathbf{1}$ of the algebra and the ideal m^2. Thus the complex vector space of derivations at φ (at m) $\mathcal{D}_\varphi = \mathcal{D}_m$ is isomorphic to m/m^2.

Thus the hypothesis (1.2) of Browder's theorem may be restated as follow

$$\dim_{\mathbb{C}} \mathcal{D}_m \,=\, n \,<\, +\infty\,.$$

1.1.1.5 C^*-algebras

A special kind of Banach algebras is that of C^*-algebras.

A $*$-*(Banach) algebra* B is a Banach algebra (over $\mathbb{C}$) endowed with an involution $* : B \to B$ (we denote $*(b)$ by b^*) such that $\forall a, b \in B$, $\forall \lambda \in \mathbb{C}$

($*$-1) $(a + b)^* = a^* + b^*$;

($*$-2) $(\lambda a)^* = \overline{\lambda} a^*$;

($*$-3) $(ab)^* = b^* a^*$;

($*$-4) $(a^*)^* = a$.

If moreover

(C^*) $\parallel a^* a \parallel = \parallel a \parallel^2$,

then B is said to be a *(Banach) C^*-algebra*.

As a consequence of the C^* condition, elements of C^*-algebras satisfy $\parallel a^* \parallel = \parallel a \parallel$ and commutative C^*-algebras are uniform algebras.

A classical example of C^*-algebra is that of $\mathbb{C}$-valued functions on a compact K, with the involution $*$ given by the conjugation.

Izrail Gelfand and Mark Naĭmark [23] proved that a commutative $*$-algebra B with unit is isometrically isomorphic to the algebra of continuous functions on its spectrum $\mathcal{C}^0(\mathfrak{M}(B))$.

A Banach algebra B of holomorphic functions can be made a $*$-algebra, by defining the involution $*$ in the following way

$$f^*(z) \,=\, \overline{f\,(\bar{z})} \quad \forall f \in B\,,$$

but cannot be made a C^*-algebra.

Since here we are mostly interested in algebras of holomorphic functions, we do not indulge in exploring C^*-algebras.

1.1.2 Fréchet algebras

A commutative algebra F over $\mathbb{C}$ with unit $\mathbf{1}$ is said to be a *Fréchet algebra* if it is a Fréchet space with a family $\{p_n\}_{n\in\mathbb{N}}$ of submultiplicative seminorms, i.e. such that

$$p_n(xy) \;\leq\; p_n(x)p_n(y)\,,$$

for all $x, y \in F$.

We may assume (up to modifying the system of seminorms with an equivalent one) that, for all $n \in \mathbb{N}$,

i) $p_n \leq p_{n+1}$;

ii) $p_n(\mathbf{1}) = 1$.

Actually giving a Fréchet algebra F is equivalent to giving a projective system of Banach algebras

$$\{B_n, \psi_n : B_{n+1} \to B_n\}_{n\in\mathbb{N}}\,,$$

the Fréchet algebra being the projective limit of the system $F = \varprojlim B_n$. For a precise definition of projective limit of a projective system, refer to [40].

1.1.2.1 Fréchet algebras: characters and spectrum

A *character* of F is a homomorphism of algebras $\chi : F \to \mathbb{C}$. The set of all characters of F is denoted by $\mathfrak{C}(F)$ It is not known whether or not all characters χ of a Fréchet algebra are continuous.

The *spectrum* of F is the set $\mathfrak{M}(F)$ of all non trivial continuous characters of F. If F^* denotes the dual space of F than

$$\mathfrak{M}(F) \;\subset\; \{b^* \in F^* \;:\, \| \, b^* \, \| \leq 1\}\,.$$

The spectrum, with the topology induced by the weak-* topology on F^* (called the *Gelfand topology*), thanks to the Banach-Alaoglu theorem and to the fact that a Fréchet algebra is the projective limit of a projective system of Banach algebras

$$\{B_n, \psi_n : B_{n+1} \to B_n\}_{n\in\mathbb{N}}\,,$$

admits a compact exhaustion by $\mathfrak{M}(B_n)$.

This implies, using the Gelfand-Mazur theorem, that a Fréchet field is isometric to $\mathbb{C}$, and hence Banach.

The fact that a Fréchet algebra F is a projective limit of Banach algebras can be used to prove that the spectrum is dense in the set of all characters:

$$\overline{\mathfrak{M}(F)} \;=\; \mathfrak{C}(F)\,.$$

Definition 1.1 Let A be an algebra of function over a topological space X. We say that X is *A-convex* if for every compact $K \subset X$, its A-hull

$$X_A \;=\; \{x \in X \mid |f(x)| \leq \max_{z\in K} |f(z)|\}\,,$$

is compact, too.

As a consequense of Banach-Steinhaus theorem and the fact that Fréchet algebras are projective limits of Banach algebras, it can be proved that

Theorem 1.4 The spectrum $\mathfrak{M}(F)$ of a Fréchet algebra F is F-convex.

Obviously this goes with the canonical Gelfand identification of $f \in F$ with its Gelfand transform $\hat{f}$. For a proof of this theorem, refer to [17, Theorems 7.32, 12.24].

1.1.2.2 Fréchet algebras: Michael's problem

Open problem. The problem whether or not all characters χ of a Fréchet algebra are continuous was first posed by Ernest Michael in 1952 (see [32]). Many attemps to solve the problem were made, based on methods of topological algebras theory.

A positive result. A first partial result was obtained in 1958 by Richard Arens: if the Fréchet algebra F is finitely rationally generated, i.e. there are a finite numer of generators $x_1, \dots, x_k \in F$ such that the subalgebra of elemnets of the form

$$\frac{P(x_1, \dots, x_k)}{Q(x_1, \dots, x_k)},$$

where P and Q are F-valued polynomials (Q is of course required to have an inverse in F) is dense in F, then all characters of F are continuous.

Reducing Micheal's problem to a specific algebra. In 1975 Dennis Clayton [12] reduced Michael's problem to solving it for a particular algebra. Let l^∞ be the space of bounded complex sequences and $\mathcal{P}$ the algebra of functions $p : l^\infty \to \mathbb{C}$ which are polynomials in (a finite number of) the coordinate functions. Introducing on $\mathcal{P}$ the semi-norms

$$p_n(p) = \sup\{|p(z)| \ : \| z \|_\infty \leq n\}$$

consider A, the completion of $\mathcal{P}$ with respect to these semi-norms. A is then a Fréchet algebra. Clayton proves that if there is a Fréchet algebra B and $\varphi : B \to \mathbb{C}$ is a discontinuous homomorphism, then there is a discontinuous homomorphism $\psi : A \to \mathbb{C}$ (i.e. the answer to Micheal's problem is negative if and only if the algebra A is a counterexample). Thus Michael's problem reduces to a problem about the algebra A.

Martin Schottenloher in 1981 [41] modified Clayton's construction to reduce Michael's problem to the same problem for a specific algebra of holomorphic functions, and similar reductions were found more recently also by Jorge Mujica [33, 34].

Even with these huge simplifications of the problem, still no answer was found.

Holomorphic dynamics approach to Michael's problem. A totally different approach to the problem was opened in 1986, when Peter Dixon and Jean Esterle [18] proved that

Proposition 1.1 Suppose there is a Fréchet algebra F with a discontinuous character. Then, for every sequence$\{s_n\}_{n\in\mathbb{N}}$ and any projective system

$$\cdots \longrightarrow \mathbb{C}^{s_{n+1}} \xrightarrow{F_n} \mathbb{C}^{s_n} \longrightarrow \cdots,$$

where all F_n are holomorphic, $\varprojlim \mathbb{C}^{s_n} \neq \emptyset$.

This result reduced proving Michael's problem to finding an example where the above projective limit is empty. In particular one gets that all characters of a Fréchet algebra are continuous if there exists a sequence of holomorphic maps $F_k : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$A = \bigcap_{k\in\mathbb{N}} F_0 \circ \cdots \circ F_k(\mathbb{C}^n) = \emptyset . \tag{1.3}$$

The best part of this result is that now both the positive and negative answer to the Michael's problem can be solved just by producing an example.

The bad part is that the example seems not easy at all to find.

Obviously if one of the F_k is constant than the intersection is not empty. If $n = 1$, by Picard's theorem, each non constant holomorphic entire function assumes all values but at most one. Hence each $F_0 \circ \cdots \circ F_k(\mathbb{C}^n)$ is a dense open subdomain of $\mathbb{C}$, hence by Baire's theorem also A is a dense open set. The example has to be found in several variables.

If $n > 1$, there are domains $\Omega \subsetneq \mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$ not dense in the whole of $\mathbb{C}^n$. This domains (called *Fatou-Bieberach domains*) could provide a way to construct the needed example.

Fatou-Bieberach domains can also be used to construct the so called *long* $\mathbb{C}^n$, which are complex manifolds of dimension n exhausted by an increasing sequence of domains each biholomorphic to $\mathbb{C}^n$. Long $\mathbb{C}^n$ can have unexpected properties (e.g. they can even have no non-constant holomorphic or plurisubharmonic functions, see [7]) and might play a role in the solution of the Michael's problem.

1.1.3 $\mathcal{LB}$ algebras and $\mathcal{LF}$ algebras

Just like one can obtain Fréchet algebras starting from a projective family of Banach algebras, one can produce new kinds of algebras starting from a directed family of Banach or Fréchet algebras.

Since, differently from Fréchet algebras, $\mathcal{LB}$ algebras and $\mathcal{LF}$ algebras do not have a nice presentation different from the one as limit, we give some details of the definition of direct limit. For a more precise definition of direct limits of directed system, refer to [40].

A topological vector space E is said to be a *locally convex space* if the origin $O \in E$ has a basis of open convex neighbourhoods whose intersection is $\{O\}$. Notice that Banach and Fréchet algebras are locally convex spaces.

Let E be a vector space, and an increasing sequence E_n of subspaces of E (i.e. such that $E_n \subset E_{n+1}$), each endowed with a topology τ_n such that E_n is a a locally convex space and the topolgy τ_{n+1} induces τ_n, and such that

$$E = \bigcup_{n\in\mathbb{N}} E_n .$$

E, endowed with the finest topology τ such that

1. (E, τ) is a locally convex space;
2. the inclusions $(E_n, \tau_n) \to (E, \tau)$ are continuous,

is called the *direct limit* of the directed system (E_n, τ_n).

An algebra which is a direct limit of Banach algebras is called a $\mathcal{LB}$ algebra, while an algebra which is a direct limit of Fréchet algebras is called a $\mathcal{LF}$ algebra.

A direct limit of complete locally convex spaces is a complete locally convex space, hence $\mathcal{LB}$ algebras and $\mathcal{LF}$ algebras are complete.

The spectrum of a $\mathcal{LB}$ algebra or $\mathcal{LF}$ algebra A is defined similarly to what has been done for Banach and Fréchet algebras:

$$\mathfrak{M}(A) = \{\varphi : A \to \mathbb{C} \,|\, \varphi \textit{is a non-trivial continuous homomorphism of algebras}\} .$$

Due to the fact that projective limits and direct limits are in duality, if A is the direct limit of the (Banach or Fréchet) algebras A_n, then its spectrum $\mathfrak{M}(A)$ is the projective limit of the spectra $\mathfrak{M}(A_n)$.

1.2 Algebras of holomorphic functions

Let D be a domain (i.e. an open connected set) in $\mathbb{C}^n$ or more generally in a complex manifold or a complex space X.

By $\mathcal{O}(D)$ we denote the algebra of holomorphic functions on D.

In this section, we will present several algebras of holomorphic functions, i.e. subalgebras of $\mathcal{O}(D)$, which fall in one of the families introduced in the previous section. These concrete algebras serve both as an example and a stimulus to study the general theory as well as a way to suggest new interesting general problems.

1.2.1 The algebra of holomorphic functions $\mathcal{O}(X)$

We start our review with the algebra of holomorphic functions on a complex manifold (or space) X.

1.2.1.1 Stein spaces and Stein manifolds

When we are interested in studying such an algebra, it is natural to consider only a certain class of complex manifolds (complex spaces), namely that of Stein manifolds (Stein spaces).

Let us show a few examples suggesting that we may want to impose some conditions on the space X.

Let X be complex torus $\mathbb{T}^n = \mathbb{C}^n/\mathbb{Z}^n$ or a complex projective space $\mathbb{CP}^n$, being X compact, all holomorphic functions on X are bounded. Since bounded holomorphic entire functions are constant, it easily follows that the only holomorphic functions on such an X are the constant functions, and the algebra $\mathcal{O}(X)$ is simply $\mathbb{C}$.

A slightly less trivial class of examples could be that of spaces X where holomorphic functions do not separate points, i.e. there exists a couple of points $x \neq y \in X$ such that $f(x) = f(y)$ for all functions $f \in \mathcal{O}(X)$. Considering the quotient space $Y = X/\sim$, where $x \sim y$ iff $f(x) = f(y)$ for all functions $f \in \mathcal{O}(X)$, then Y is a complex space such that $\mathcal{O}(Y) = \mathcal{O}(X)$ and where holomorphic functions separate points of Y.

Another (less immediate) condition we may want to ask is that of X being holomorphically convex, i.e. such that for any compact $K \Subset X$ its holomorphic hull

$$\hat{K}_{\mathcal{O}(X)} = \{x \in X \mid |f(x)| \leq \max_{z \in K} |f(z)|, \ \forall f \in \mathcal{O}(X)\}$$

is also compact. This is exactly the notion of A-convexity given in much more generality in Definition 1.1, when $A = \mathcal{O}(X)$. Such a condition prevents the *Hartogs phenomenon* to happen, i.e. X is the natural space for considering the algebra of functions $\mathcal{O}(X)$ and is not a subspace of a bigger complex space X with the same algebra. The Hartogs phenomenon is a phenomenon characteristic of several complex variables and does not happen in one complex variable, discovered by Fritz Hartogs in 1906 [27].

Thus, if we are interested in studying the algebra of holomorphic functions on a complex space X, we may limit ourselves to spaces such that

- (S_1) holomorphic functions separate points of X;
- (S_2) X is holomorphically convex.

A complex manifold (resp. complex space) X satisfying (S_1) and (S_2) is called a *Stein manifold* (resp. *Stein space*). A subdomain D of a Stein space X is said to be *locally Stein* if for every point $x \in bD$ there is a neighbourhood $V \ni x$ such that $V \cap D$ is Stein.

For a Stein manifold X of dimension n it holds a result very similar to the Whitney embedding theorem for real manifolds: they can be properly embedded in $\mathbb{C}^{2n+1}$.

For a review on Stein manifolds and spaces, we refer the reader to [21, Ch. 2, 3] and [17, Ch. 11].

1.2.1.2 The spectrum of the algebra of holomorphic functions on a Stein space

Let X be an n-dimensional complex space. Denote by

$$\delta : X \to \mathfrak{M}(\mathcal{O}(X))$$

the natural map which associates $x \in X$ to the maximal ideal

$$M_x = \{f \in \mathcal{O}(X) \mid f(x) = 0\}.$$

$\mathcal{O}(X)$ is a Fréchet algebra when endowed with the sup-seminorms on an increasing exhaustion of compact.

A Stein space (manifold) is completely determined by its algebra of holomorphic function, as proved in great generality by Constantin Bănică and Octavian Stănăşilă in 1969 [4].

Theorem 1.5 A complex space X is Stein iff δ is a homeomorphism. Moreover, for every complex space Y

$$Hol(Y, X) \simeq Hom_{\mathcal{C}^0}(\mathcal{O}(X), \mathcal{O}(Y)).$$

In particular, two Stein spaces are biholomorphic iff their Fréchet algebras of holomorphic functions are isomorphic.

1.2.1.3 The envelope of holomorphy problem

As we discussed above, there are some complex spaces X which are not the natural domains for holomorphic functions, i.e. an Hartogs phenomenon happens and all holomorphic functions on X extend (uniquely thanks to the identity principle) to a bigger domain. This does not happen if X is Stein.

Hence, a natural problem arises. An *envelope* (or *hull*) *of holomorphy* of a complex space (or manifold) X is a Stein space $\hat{X}$ and a holomorphic open embedding $j : X \to \hat{X}$ such that $j^* : \mathcal{O}(\hat{X}) \to \mathcal{O}(X)$ is an isomorphism of Fréchet algebras, i.e. all holomorphic functions on X extend uniquely to the Stein space $\hat{X}$.

Since $X \subset \hat{X}$, and $\hat{X}$ is required to be Stein, an obvious necessary condition for X to have an envelope of holomorphy is to have holomorphic functions that separate its points.

Notice that, thanks to the previous theorem, if an envelope of holomorphy exists, it is unique up to biholomorphisms.

Moreover notice that, even if X is a complex manifold, its envelope of holomorphy may be a Stein space, and not a manifold. Indeed, just consider a Stein space $\hat{X}$ (with singular set $Z \neq \emptyset$). Then $X = \hat{X} \setminus Z$ is a complex manifold, which has the Stein space $\hat{X}$ as envelope of holomorphy, where $j : X = \hat{X} \setminus Z \to \hat{X}$ is the inclusion.

Thanks to Theorem 1.5, if a complex space X admits an envelope of holomorphy $\hat{X}$, then its envelope —being a Stein space— is completely determined up to biholomorphisms by its algebra of holomorphic functions $\mathcal{O}(\hat{X}) \simeq \mathcal{O}(X)$. Hence, the natural candidate for $\hat{X}$ is the spectrum $\mathfrak{M}(\mathcal{O}(X))$. Thus the envelope of holomorphy problem may be restated in the following way.

Envelope of holomorphy problem: let X be a complex space such that $\mathcal{O}(X)$ separates points of X. Give the spectrum $\mathfrak{M}(\mathcal{O}(X))$ a complex structure such that

1. the natural map $\delta : X \to \mathfrak{M}(\mathcal{O}(X))$ is an open holomorphic embedding;
2. all the Gelfand transforms $\hat{f}$, $f \in \mathcal{O}(X)$, are holomorphic functions on the spectrum.

We remark that $\mathfrak{M}(\mathcal{O}(X))$ is a Stein space thanks to condition (1) and the fact that $\mathfrak{M}(\mathcal{O}(X))$ is $\mathcal{O}(\hat{X})$-convex, as it follows from the general Theorem 1.4 on Fréchet algebras.

Using this characterization of the envelope of holomorphy problem, one usually say that $\mathcal{O}(X)$ is a *Stein algebra* iff $\mathcal{O}(X) \simeq \mathcal{O}(\hat{X})$ for a Stein space $\hat{X}$ or —equivalently— if $\mathfrak{M}(\mathcal{O}(X))$ can be given a complex structure satisfying (1) and (2) above.

The envelope of holomorphy problem: positive results. The envelope of holomorphy problem has a positive answer for a huge class of domains. Henri Cartan, Peter Thullen and Kiyoshi Oka gave the positive answer for all Riemann domains X over $\mathbb{C}^n$, i.e. domains X with a holomorphic projection $\pi : X \to \mathbb{C}^n$ which is locally a biholomorphism. For a presentation of the proof, we refer to the paper by Hugo Rossi [38]. The classical proofs by Cartan, Thullen and Oka use classical arguments of the theory of several complex variables. Their results were also obtained using a different strategy of proof, which used abstract theory of algebras of functions —namely, interpolating semi-norms, by Erret Bishop in 1963 [6].

The approach of Bishop solves more generally the problem of the envelope of holomorphy for Riemann domains D relative to a subalgebra $\mathcal{A} \subset \mathcal{O}(D)$.

The same is true for Riemann domains X over a Stein manifold Y, i.e. domains X with a holomorphic projection $\pi : X \to Y$ which is locally a biholomorphism. In this greater generality the result as proven in 1960 by Ferdinand Docquier and Hans Grauert [15].

In particular, all domains $D \subset \mathbb{C}^n$ admit an envelope of holomorphy (since subdomains are a trivial examples of Riemann domains). It is nevertheless worth noticing that if $X \subset Y$, where Y is a Stein manifold, its envelope of holomorphy needs not to be contained in Y. This is false even when $Y = \mathbb{C}^n$. Henri Cartan pointed out a counterexample for $Y = \mathbb{C}^2$. Obviously there is no counterexample for $Y = \mathbb{C}$, since all domains in $\mathbb{C}$ are holomorphically convex, hence Stein.

The envelope of holomorphy problem: negative results. To construct a domain for which the envelope of holomorphy does not exist is quite a complex and technical task. A first example was found by Hans Grauert in 1963 [25]. A clear exposition of the example, which is a domain with a smooth boundary, almost everywhere strongly Levi-convex, can be found in [30].

As we've already noted, subdomains of Stein spaces always admit an envelope of holomorphy. This is no longer the case for subdomains of Stein spaces, as proved by Mihnea Colţoiu and Klas Diederich in 2000 [13]. Let X be a Stein space and $D \subset X$ an open set. D is said to verify the *strong hypersection condition* (SHSC) if for every open set $V \supset D$ and every closed complex hypersurface $H \subset V$ then

$H \cap D$ is Stein.

Theorem 1.6 Let D be a relatively compact open set of a Stein space X satisfying one of the following conditions:

- D is locally Stein;
- D is an increasing union of Stein spaces;
- D satisfies SHSC.

Then either D is Stein or it does not have an envelope of holomorphy.

In [13] it is constructed a Stein space X of pure dimension 3 and a closed analytic surface $A \subset X$ such that $D = X \setminus A$ satisfies SHSC and is not Stein, hence does not have an envelope of holomorphy.

It is worth citing the fact that the other two hypothesis in the above theorem are linked with some open problems, namely the Levi problem and the union problem.

Open problem: *Levi problem.* Is a locally Stein open subset of a Stein space Stein?

The answer is known to be positive for $X = \mathbb{C}^n$ (Fifties: Oka [36], Bremermann [8], Norguet [35]), X a Stein manifold (1960: Docquier and Grauert [15]), or even X a Stein space with isolated singularities (1964: Andreotti and Narasimhan [2]). For a survey of the Levi problem, refer to [44].

Open problem: *union problem.* Is an increasing union of Stein subdomains D of a Stein space X Stein?

The answer is known to be positive for $X = \mathbb{C}^n$ (1939: Behnke and Stein [5]) or X a Stein manifold (1960: Docquier and Grauert [15]). The general problem is unsolved even for isolated singularities. If X is a normal Stein space, D is known to be a domain of holomorphy, i.e. for each point $x \in bD$ there is a function unbounded near x. Stein spaces are domain of holomorphy, but the converse fails to be true if the dimension of the space is at least 3.

Other notions of envelope. The notion of envelope is linked to the choice of an algebra of functions. It is worth noticing that in complex geometry, this notion can be generalized to other families of analytic objects, as divisors, principal divisors or analytic subsets. Since this survey is on algebras of functions, we do not spend time of these more general notions.

While all domains in $\mathbb{C}^n$ are Stein and hence coincide with their envelopes, if we consider small algebras things may get trickier. Consider e.g.

$$D = \mathbb{C} \setminus \{z \in R \,|\, z \leq 0\}\,,$$

and the algebra $\mathcal{A}$ generated by the holomorphic function $\log z$. Then D is not $\mathcal{A}$-convex and its $\mathcal{A}$-envelope is the infinite layer Riemann spiral projecting over $\mathbb{C} \setminus \{0\}$.

1.2.2 The algebra of bounded holomorphic functions $H^\infty(D)$

We denote by $H^\infty(D)$ the algebra of bounded holomorphic functions on D. $H^\infty(D)$ is a Banach algebra when endowed with the sup-norm:

$$\| f \|_\infty = \sup_{z \in D} |f(z)| .$$

The notation $H^\infty(D)$ is due to the fact that $H^\infty(D) = \mathcal{O}(D) \cap L^\infty(D)$ and the norm on $H^\infty(D)$ is just the restriction of the sup-norm on the Banach algebra $L^\infty(D)$.

The space $H^\infty(D)$ was firstly considered in the case $D = \Delta$, the unit disk of $\mathbb{C}$. In this situation, also the spaces $H^p(\Delta)$, $0 < p \leq \infty$ are defined. The spaces $H^p(\Delta)$ are usually known as Hardy spaces of the disk. For a nice treaty on these spaces, refer to the classical book of Peter Duren [19].

Let $0 < p < \infty$. A holomorphic function f on Δ is said to be in $H^p(\Delta)$ if

$$\|f\|_p^p = \sup_{0<r<1} \int_0^1 |f(re^{2\pi i\theta})| \, d\theta < \infty$$

For $1 \leq p \leq \infty$, $H^p(\Delta)$ endowed with the norm $\| \cdot \|_p$ is a Banach algebras, and for $p = 2$ the Banach norm is induced by a Hermitian product, turning $H^2(\Delta)$ in a complex Hilbert space of functions.

Assume $H^\infty(D)$ separates the points of D, i.e. given any two points $z \neq w \in D$ there is a function $f \in H^\infty(D)$ such that $f(z) \neq f(w)$.

Then there is a natural embedding ι of the domain D into the (maximal) spectrum $\mathfrak{M}(H^\infty(D))$ given by

$$\iota(z) = \{f \in H^\infty(D) \mid f(z) = 0\} ,$$

i.e. z is sent to the ideals of functions vanishing in z.

Obviouly $\iota(D)$ is not all of the spectrum (e.g. the ideals of functions vanishing at a point of the boundary is not in the image), also for the compactness of the spectrum.

Consider the set

$$\mathfrak{M}(H^\infty(D)) \setminus \overline{\iota(D)} ,$$

called the *corona*.

The *corona conjecture* states that the corona is empty, i.e. the domain D naturally embeds densely in the spectrum of $\mathfrak{M}(H^\infty(D))$.

We remark that the corona conjecture has an analytic equivalent form:

Theorem 1.7 Assume bounded holomorphic functions separate points of D.

$\iota(D)$ is dense in $\mathfrak{M}(H^\infty(D))$ if and only if for all $f_1, \ldots, f_n \in H^\infty(D)$ such that

$$\sum_{i=1}^n |f_i(z)|^2 \geq \delta > 0$$

for all $z \in D$, there exists $g_1, \ldots, g_n \in H^\infty(D)$ such that

$$\sum_{i=1}^{n} f_i(z) g_i(z) \;\equiv\; 1 \quad \forall z \in D.$$

The corona conjecture has a positive answer for the unit disk $\Delta \subset \mathbb{C}$ (first proof due to Lennart Carleson in 1962 [10]), while the answer is negative in general (for a counterexample, see e.g. the one by Nessim Sibony [42]) and is still an open problem for simple domains as the unit ball and unit polidisk in $\mathbb{C}^n$, $n > 1$.

In the disk the corona conjecture is deeply linked with the study of Carleson measures for $H^p(\Delta)$, i.e. measures μ on Δ such that the inclusion $H^p(\Delta) \to L^p(\mu)$ is continuous.

A class of algebras of functions strongly linked to the Hardy algebras is that of Bergmann algebras of functions:

$$\mathcal{A}^p(D) \;=\; \mathcal{O}(D) \cap L^p(D)\,,$$

considered as a subalgebra of $L^p(D)$. In Bergmann algebras a central problem is that of the characterization of Carleson measures, while there are no problems directly linked to the theory of topological algebras, hence we do not analyze them here.

To have a broader review on the corona problem and on Carleson measures of Hardy and Bergmann spaces, refer to [16] and [39].

1.2.3 The algebras of holomorphic functions continuous (or more) up to the boundary $A^k(D)$

Let $D \subset \mathbb{C}^n$ be a domain. For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $A^k(D) = \mathcal{C}^k(\overline{D}) \cap \mathcal{O}(D)$, i.e. the algebra of functions holomorphic on D and of class $\mathcal{C}^k$ up to the boundary. Since $A^k(D)$ is a closed subalgebra of $\mathcal{C}^k(\overline{D})$, it is a Banach algebra if D is bounded and $k \in \mathbb{N}$, when endowed with the norm

$$\| \, f \, \|_{A^k(D)} \;=\; \sup_{j \le k, z \in \overline{D}} |f^{(j)}(z)|\,.$$

In case either D is unbounded or $k = \infty$ (or both), then $A^k(D)$ is a Fréchet algebra, endowed with the seminorms ($1 \le \ell \in \mathbb{N}$, $z_0 \in \overline{D}$ fixed)

$$p_\ell(f) \;=\; \| \, f_{|_{B_\ell(z_0) \cap \overline{D}}} \, \|_{A^\ell(B_\ell \cap \overline{D})}\,,$$

where $B_\ell(z_0) = \{z \in \mathbb{C}^n \mid |z - z_0| < \ell\}$.

Convexity properties of the domain D (plain convexity, or Levi-convexity at the points of the boundary) result in interesting properties of the spectrum of $A^k(D)$.

Theorem 1.8 Let $D \Subset \mathbb{C}^n$ be a convex bounded domain (or a convex domain with smooth strongly Levi-convex boundary. Then

$$\mathfrak{M}(A^k(D)) \;=\; \overline{D}\,,$$

for all $k \in \mathbb{N} \cup \{\infty\}$.

The main ingredient in the proof is that, under the hypothesis, functions in $A^k(D)$ can be approximated by holomorphic functions in a system of Stein neighbourhoods whose intersection is $\overline{D}$. Since for any Stein manifold X closed ideals of the algebra of holomorphic functions correspond to the points of X, it follows the thesis of the theorem.

A different way of proof, based on the $\mathcal{C}^\infty$-regularity for the $\bar{\partial}$-problem on $\overline{D}$ (result due to Joseph Kohn [31]), allows the domain D to have smooth Levi-convex boundary, i.e. to drop the strong Levi-convexity of the boundary. This was estabilished by Monique Hakim and Nessim Sibony in 1980 [26] and further generalized by Giuseppe Tomassini in 1983 [46] to the case when the domain D is no long required to be bounded.

All these results hold (with the same proof) to domains D in a complex manifold X such that there exists a non-constant subharmonic function defined on a neighbourhood of bD.

Theorem 1.9 Let X be a complex manifold and $D \subset X$ a domain such that there exists a non-constant subharmonic function defined on a neighbourhood of bD. Then

$$\mathfrak{M}(A^k(D)) = \overline{D},$$

for all $k \in \mathbb{N} \cup \{\infty\}$.

1.2.3.1 Šilov boundary of $A^k(D)$

If $D \Subset \mathbb{C}^n$ is a bounded domain and $k \in \mathbb{N}$, then $A^k(D)$ is a Banach algebra whose spectrum is $\mathfrak{M}(A^k(D)) = \overline{D}$.

Due to the maximum principle for holomorphic functions, the topological boundary of $\overline{D}$ is a boundary for the algebra $A^k(D)$. Actually, the name *boundary* for Banach algebras derives exactly from this remark. Hence the Šilov boundary of the algebra is a subset of the topological boundary of $\overline{D}$:

$$\gamma\mathfrak{M}(A^k(D)) \subset b\overline{D}.$$

Depending on D, however, the Šilov boundary may or may not coincide with the whole topological boundary.

If D is strictly convex, then for every point of the boundary z_0 there is a (complex) hyperplane $\{L = 0\}$ through z_0 touching $\overline{D}$ only in z_0. Thus $f = 1/L$ is a function holomorphic on $\mathbb{C}^n \setminus \{z_0\}$. A small deformation of this function gives a function holomorphic in D and even $\mathcal{C}^\infty$ up to the boundary, which attains its maximum at z_0. Hence in this case the Šilov boundary coincides with the topological boundary.

The above argument may be carried on also if D is supposed to be strongly pseudoconvex, or even for other notions of convexity. As it should be clear, hypothesis of convexity on the domain, imply several nice properties on algebras of holomorphic functions (see [1]).

On the opposite side of the spectrum, where the boundary has big flat parts, let us suppose D is a product domain, i.e.

$$D = D_1 \times \cdots \times D_h ,$$

where $D_j \subset \mathbb{C}^{n_j}$ and $n_1 + \cdots + n_h = n$.

Then the Šilov boundary of the algebra $A^k(D)$ is the cartesian product of the Šilov boundaries of the algebras $A^k(D_j)$, hence it is strictly contained in the topological boundary of $\overline{D}$.

It is worth noticing that, while the real dimension of the topological boundary is $2n - 1$ (if the topological boundary is a manifold), the real dimension of the Šilov boundary (again, if it is a manifold) is lower, for product domains.

In the particular case of a polidisk

$$D = \Delta_1 \times \cdots \times \Delta_n ,$$

$\Delta_k \subset \mathbb{C}$ being disks, the Šilov boundary of $A^k(D)$ is exactly the product of the topological boundaries of the disks, i.e. a real torus of dimension n.

1.2.3.2 Finitely generated maximal ideals

Let now $D \Subset \mathbb{C}^n$ be a bounded domain with a (sufficiently) smooth Levi-convex boundary, $n > 1$. Let moreover $k \in \mathbb{N}$, so that $A^k(D)$ is a Banach algebra. As already noted, $\mathfrak{M}(A^k(D)) = \overline{D}$.

Remark 1.1 In this situation, Gleason's theorem (theorem 1.2) implies that finitely generated ideals have a structure of a complex space. This implies that points of $bD \subset \mathfrak{M}(A^k(D))$ are not finitely generated maximal ideals. Indeed, if $z \in bD$ was a finitely generated maximal ideal, there would exist a neighbourhood $z \in U \subset \bar{D}$ with a complex space structure. In particular $U \cap bD$ would locally disconnect U, which is a contradiction.

It remains to determine whether the points $a \in D \subset \mathfrak{M}(A^k(D))$ are finitely generated maximal ideals or not. This is known as the *Gleason problem*.

Given a point $a = (a_1, \ldots, a_n) \in D \Subset \mathbb{C}^n$, a natural candidate set of generators of the maximal ideal M_a is given by the coordinate functions $z_j - a_j$, $j = 1, \ldots, n$. They actually are a set of generators, provided the boundary bD is of class at least $\mathcal{C}^{k+3}$ and D is convex, as proved by Gennadiĭ Henkin in 1971 [29].

Theorem 1.10 Let $D \Subset \mathbb{C}^n$ be a bounded convex domain with boundary bD is of class at least $\mathcal{C}^{k+3}$ ($k \in \mathcal{N}$) and $a = (a_1, \ldots, a_n) \in D$. Then the maximal ideal M_a of $A^k(D)$ is finitely generated by the coordinate functions $z_j - a_j$, $j = 1, \ldots, n$.

The quite elementary proof explicitely constructs, for $f \in M_a$, functions $h_j \in A^k(D)$, $j = 1, \ldots, n$, via an integral on the segment joining a and a generic point $z \in \bar{D}$, such that

$$f(z) = \sum_{j=1}^{n} h_j(z)(z_j - a_j) .$$

It is worth noticing that since $A^\infty(D)$ is not a Banach algebra (not even when D is bounded), remark 1.1 does not apply in this case. It is indeed possible, with the very same proof, to prove the stronger

Theorem 1.11 Let $D \subset \mathbb{C}^n$ be a (non necessarily bounded) convex domain with boundary bD of class $\mathcal{C}^\infty$ and $a = (a_1, \ldots, a_n) \in D$. Then the maximal ideal M_a of $A^\infty(D)$ is finitely generated by the coordinate functions $z_j - a_j$, $j = 1, \ldots, n$.

Dropping the convexity property in favour of strong Levi-convexity at all points of the boundary (even if a greater smoothness of the boundary is required, mainly for technical reasons), both results remains true, altough the proof is no longer constructive and heavily uses sheaf theory. The following theorems were estabilished by Giuseppe Tomassini in 1983 [46].

Theorem 1.12 Let $D \Subset \mathbb{C}^n$ be a bounded domain with strongly Levi-convex boundary bD of class $\mathcal{C}^\infty$ and $a = (a_1, \ldots, a_n) \in D$. Then the maximal ideal M_a of $A^k(D)$ is finitely generated by the coordinate functions $z_j - a_j$, $j = 1, \ldots, n$.

Theorem 1.13 Let $D \subset \mathbb{C}^n$ be a (non necessarily bounded) domain with strongly Levi-convex boundary bD of class $\mathcal{C}^\infty$ and $a = (a_1, \ldots, a_n) \in D$. Then the maximal ideal M_a of $A^\infty(D)$ is finitely generated by the coordinate functions $z_j - a_j$, $j = 1, \ldots, n$.

1.2.4 The algebra of functions holomorphic in a neighbourhood of a compact $\mathcal{O}(K)$

Let X be a complex space (usually a Stein space or even $\mathbb{C}^n$)and $K \subset X$ a closed set (possibly a compact). Consider the algebra $\mathcal{O}(K)$ of germs of holomorphic functions, i.e. the algebra of functions holomorphic on an open neighbourhood of K. Considering a family of open neighbourhoods U_k of K, each one contained in the next, and whose intersection is K, $\mathcal{O}(K)$ is easily seen to be an $\mathcal{LF}$-algebra, being the direct limit of the Fréchet algebras $\mathcal{O}(U_k)$.

If K is the closure of a bounded domain in $\mathbb{C}^n$ or in a Stein space X, the structure of the spectrum $(\mathcal{O}(K))$ (which for obvious reasons is called the *holomorphic envelope* of K) was studied by Reese Harvey and Raymond Wells at the end of the Sixties [28, 48].

Regularity hypothesis on the set K give interesting algebraic properties of the algebra $\mathcal{O}(K)$. Indeed

Theorem 1.14 If K is a compact subset of a connected normal space X, then $\mathcal{O}(K)$ is an integrally closed ring.

Theorem 1.15 (Frisch 1965 [22], Siu 1969 [43]) If K is a semianalytic compact set which has a fundamental system of Stein neighbourhoods in a complex space X, then $\mathcal{O}(K)$ is a noetherian ring.

Theorem 1.16 (Dales 1974 [14]) If K is a contractible semianalytic compact set which has a fundamental system of Stein neighbourhoods in a locally factorial complex space X, then $\mathcal{O}(K)$ is a factorial ring.

1.2.5 The algebra of holomorphic functions with polynomial growth $\mathcal{P}(D)$

Let X be a complex space endowed with a metric, and $D \subsetneq X$ a domain. A function $f : D \to \mathbb{C}$ of class $\mathcal{C}^\infty$ is said to have a *polynomial growth* if for every K compact subset of X there exists $m \in \mathbb{N}$ such that

$$\sup_{z \in K \cap D} d(z, bD)^m \, |f(z)| \, ,$$

$d(z, bD)$ being the distance of z from the boundary of D. The same definition may of course be given also for differential forms.

The name polynomial growth is due to the fact that, if $X = \mathbb{CP}^n$ with the Fubini-Study metric and $D = \mathbb{C}^n$, then the holomorphic functions with polynomial growth are exactly the polynomials.

The holomorphic functions with polinomial growth on D form an algebra, denoted with $\mathcal{P}(D)$. Note that actually the algebra depends not only on D but also on the ambient space X and its metric. We do not stress these in the name, as the only result we cite here is in $\mathbb{C}^n$ with the euclidean metric.

Let $X = \mathbb{C}^n$, with the euclidean metric, and $D \subsetneq \mathbb{C}^n$ be a domain with a smooth strongly Levi-convex boundary. In this situation, Paolo Cerrone and Giuseppe Tomassini proved in 1984 [11] some theorems of finiteness for ideals of functions of polynomial growth, leading to an interesting result on the $\bar{\partial}$-problem.

Theorem 1.17 Let $D \subsetneq \mathbb{C}^n$ be a domain with a smooth strongly Levi-convex boundary and η a $\mathcal{C}^\infty$-smooth (p, q)-form with polynomial growth. Then the equation

$$\bar{\partial}\mu = \eta$$

has a solution μ, $\mathcal{C}^\infty$-smooth $(p, q-1)$-form with polynomial growth.

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