Some recent contributions to CR-submersions

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In this survey work\(^1\) we give some our recent contributions on submersions of CR-, or QR-submanifolds, as well as some results of other authors on the topic.

The study of Riemannian submersions was initiated by B. O'Neill ([21]) and A. Gray ([13]). This theory was developed very much in the last thirty five years. A good reference is chapter 9 of Besse’s book [7]; see also some recent papers [1], [2].

We recall that a Riemannian submersion yields a vertical distribution \(\mathcal{V}\) which is integrable and a horizontal distribution \(\mathcal{H}\) (see [7], p. 236). On the other hand, on a CR-submanifold \(M\) of a Kähler manifold \((\tilde{M}, \tilde{g}, J)\) there are two orthogonal complementary distributions \(\mathcal{V}\) and \(\mathcal{H}\), such that \(\mathcal{H}\) is \(J\)-invariant and \(\mathcal{V}\) is totally real (cf. A. Beyancu [5]).

Recently, S. Kobayashi considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kähler manifold in terms of distributions ([15]).

Let \(TM^\perp\) be the normal bundle of \(M\) in \(\tilde{M}\). We denote by \(\mu\) the orthogonal complementary vector bundle to \(J(\mathcal{V})\) in \(TM^\perp\), i.e. \(TM^\perp = J(\mathcal{V}) \oplus \mu\). It is clear that \(\mu\) is a holomorphic sub bundle of \(TM^\perp\), i.e. \(J(\mu) = \mu\).

**Definition 1.** Let \(M\) be a CR-submanifold of a Kähler manifold \((\tilde{M}, g, J)\). A CR-submersion from a CR-submanifold \(M\) onto an almost Hermitian manifold \((M', g', J')\) is a Riemannian submersion \(\pi : M \to M'\), such that:

\[(i)\] \(\mathcal{V}\) is the kernel of \(\pi_*\);

\[(ii)\] for each \(x \in M\), \(\pi_* : \mathcal{H}_x \to T_{\pi(x)}M'\) is a complex isometry, i.e. \(\pi_* \circ J = J' \circ \pi_*\).

The above definition is given by S. Kobayashi in the case where \(\mu\) is a null sub bundle of \(TM^\perp\) (see [15]). If \(J\mathcal{V}_x = T_xM^\perp\), for any \(x \in M\) we say that \(M\) is a *generic CR-submanifold* of \(M\) (cf. [23]). For example, any real orientable hypersurface of \(M\) is a generic CR-submanifold of \(M\).
The vertical distribution $\mathcal{V}$ of a Riemannian submersion is the kernel of $\pi^*$, so that $\mathcal{V}$ is an integrable distribution. D.B. Blair and B.Y. Chen have proved that any totally real distribution $\mathcal{V}$ of a CR-submanifold $M$ of a Kähler manifold $\tilde{M}$ is always integrable ([3]).

We have the first basic result:

**Theorem 1.** Let $M$ be a CR-submanifold of a Kähler manifold $\tilde{M}$ and let $\pi : M \to M'$ be a CR-submersion of $M$ onto an almost Hermitian manifold $M'$. Then $M'$ is a Kähler manifold.

This theorem is proved for the generic case $\mathcal{V} = \{0\}$ in [15] and for the case $\mathcal{V} \neq \{0\}$ in [16].

In the generic case, another contribution was given in [15] on the relation between the holomorphic sectional curvatures of $\tilde{M}$ restricted to $\mathcal{H}$ and those of $M'$. Namely, one has shown the following formula:

$$
\tilde{K}(X) = K'(\pi_*X) - 4\|B(X, X)\|^2,
$$

for any unit horizontal vector $X$, where $\tilde{K}$ and $K'$ are the holomorphic sectional curvatures of $\tilde{M}$ and $M'$ respectively, and $B$ is the second fundamental form of $M$ in $\tilde{M}$.

Now we will present our investigations on the CR-submersions from an extrinsic hypersphere of an Einstein-Kähler manifold. We say that a totally umbilical hypersurface $M$ of a Riemannian manifold $\tilde{M}$ is an extrinsic hypersphere iff the mean curvature vector field $H$ is parallel and $H_x \neq 0$, for any $x \in M$. Many of the basic results concerning extrinsic spheres in Riemannian and Kählerian geometry were obtained by B.Y. Chen ([8]). We have,

$$
B(E, F) = g(E, F)H,
$$

for any couple of vector fields $E, F$ on $M$. If we put $k = \|H\|$ (where the norm $\|\cdot\|$ is respect to scalar product induced by $g$ on every tangent space to $M$), then $\xi = -\mathcal{J}N$ is a global unit vector on $M$.

We see that $M$ is a CR-hypersurface of $\tilde{M}$ such that $\mathcal{V}$ is the one dimensional anti-invariant distribution generated by the vector field $\xi$.

Then, in [16] we proved the following theorem:

**Theorem 2.** Let $M$ be an orientable extrinsic hypersphere of a Kähler-Einstein manifold $\tilde{M}$. If $\pi = M \to M'$ is a CR-submersion
of $M$ onto an almost Hermitian manifold $M'$, then $M'$ is a Kähler-Einstein manifold.

Now we suppose that $\tilde{M}$ is a complex space form. Then we may state (see [9]):

**Theorem 3.** Let $\pi : M \to M'$ be a CR-submersion of a totally umbilical CR-submanifold ($\dim M \geq 5$) of a complex space form $\tilde{M}(c)$ onto an almost Hermitian manifold $M'$. Then $M'$ is also a complex space form.

A CR-submanifold $M$ of a Kähler manifold $\tilde{M}$ is said to be a mixed foliate if $\mathcal{H}$ is an integrable distribution and $B(U, X) = 0$ for any $U \in \mathcal{V}_x$, $X \in \mathcal{H}_x$, $x \in M$. In [9] the authors proved the following result.

**Theorem 4.** Let $M$ be a mixed foliate CR-submanifold of a Kähler-Einstein manifold $\tilde{M}$. If $\pi : M \to M'$ is a CR-submersion of $M$ onto an almost Hermitian manifold $M'$, then $M'$ is also a Kähler-Einstein manifold.

**Remarks.**

An extrinsic hypersphere of a Kähler-Einstein manifold $\tilde{M}$ is not a mixed foliate CR-submanifold (see Th. 2.).

In [10], the authors studied similar problems for CR-submanifolds in Hermitian, quasi-Kähler or nearly Kähler manifolds. In this cases, totally real distributions are not necessarily integrable. To overcome this difficulty the authors consider the submersions $\pi : M \to M'$ of CR-submanifolds $M$ with an integrable distribution $\mathcal{V}$ onto an almost Hermitian manifold $M'$. For example, a real hypersurface in $S^6$ (which is a nearly Kähler manifold) is a CR-hypersurface with a 1-dimensional distribution $\mathcal{V}$ which is always integrable.

Now we describe some results obtained by F. Narita ([20]). Let $M$ be a locally conformal Kähler manifold and let $L$ be the Lee vector field on $M$ (cf. [11]). Then we have

**Theorem 5.** Let $M$ be a generic CR-submanifold of a locally conformal Kähler manifold $\tilde{M}$ and let $\pi : M \to M'$ be a CR-submersion of $M$ onto an almost Hermitian manifold. Then the Lee vector $L$ belongs to $\mathcal{H} \oplus T M^\perp$ and for any horizontal unit vector $X \in \mathcal{H}$ we have
where $A$ is the integrability tensor with respect to $\pi$.
Moreover, if we assume in addition that the horizontal component $hL$ of $L$ is basic and $\dim M' \geq 4$, then $M'$ is also a locally conformal Kähler manifold. In particular, if $M$ is a generalized Hopf manifold and if the Lee vector $L$ is basic, then $M'$ is also a generalized manifold.

Finally, the concept of CR-submersion was extended to semi-invariant (or contact CR-manifold) submanifolds in the Sasakian geometry by N. Papaghiuc ([22]). He obtained basic properties of CR-submersion of a semi-invariant submanifold of a Sasakian manifold onto an almost contact manifold and he studied various relations between the sectional curvatures of $\tilde{M}$ and $M'$.

In the last part of this survey paper we refer to some Riemannian submersions from a hypersurface of a quaternionic Kähler manifold. First, we recall some definitions.

We say that a $4(m+1)$-dimensional manifold $\tilde{M}$ with a metric $g$ is a quaternionic Kähler manifold ($m \geq 1$), if there exists a 3-dimensional vector bundle $\mathcal{V}$ of tensors of type $(1, 1)$ on $\tilde{M}$ satisfying the following conditions:

a) In any coordinate neighborhood $\tilde{U}$ on $\tilde{M}$ there is a local basis with almost Hermitian structures $\{ J_a, g \}$ such that $J_a^2 = -1d$, $a \in \{1, 2, 3\}$ and $J_a \circ J_b = -J_b \circ J_a = J_c$, for any cyclic permutation $(a, b, c)$ of $(1, 2, 3)$.

b) For any local section $\varphi$ of $\mathcal{V}$ and any tangent vector $X$ to $\tilde{M}$, $\nabla_X \varphi$ is also a local section in $\mathcal{V}$, where $\nabla$ denotes the Levi-Civita connection on $\tilde{M}$ ([14], [17]).

Let $M$ be an orientable hypersurface of $\tilde{M}$ and le $\xi$ be a unit normal field defined on $M$. One obtains a distribution $\mathcal{V}$ on $M$ which is locally represented by $\{ \xi_a \}$, $1 \leq a \leq 3$, on $\tilde{U}$, where $\xi_a = -J_a(\xi)$, $a \in \{1, 2, 3\}$. Let $\mathcal{H}$ be the orthogonal complementary distribution to $\mathcal{V}$ with respect to $g$ on $M$.

We say that $M$ is a QR-hypersurface of $\tilde{M}$ (cf. [6]). Firstly, we remark that a real hypersurface of a quaternionic Kähler manifold is not a CR-hypersurface of $\tilde{M}$.

**Definition 2.** Let $M$ be a QR-hypersurface of a quaternionic Kähler manifold $\tilde{M}$, such that the vertical distribution $\mathcal{V}$ is integrable.
We say that a Riemannian submersion $\pi : M \to M'$ is a QR-submersion if the following conditions are satisfied:

i) $V$ is the kernel of $\pi^*$;

ii) for each $x$, $\pi_* : \mathcal{H}_x \to T_{\pi(x)}M'$ is an isometry with respect to each complex structure of $\mathcal{H}_x$ and $T_{\pi(x)}M'$.

It is well known that the vertical distribution $\mathcal{V}$ is integrable if and only if $B(U, X) = 0$ for any $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. In this case we say that $M$ is a mixed geodesic QR-hypersurface.

We proved the following result ([17]).

**Theorem 6.** Let $M$ be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $\tilde{M}$. If $\pi : M \to M'$ is a QR-submersion of $M$ on an almost quaternionic Hermitian manifold, then $M'$ is a quaternionic Kähler manifold.

**Theorem 7.** Let $M$ be a totally umbilical, but not totally geodesic, QR-hypersurface of a quaternionic Kähler manifold $\tilde{M}$. Then,

a) $\tilde{K}(U, V) = K(U, V) - \|H\|^2$, where $\{U, V\}$ is an orthonormal basis of the vertical 2-plane $\alpha$, $\alpha \in \mathcal{V}_x$, $x \in M$ and $\tilde{K}, K$ denote the sectional curvatures of $\alpha$ on $\tilde{M}, M$, respectively.

b) $K(X, Y) = K'(X', Y') - 3\|H\|^2 \sum_{a=1}^{3} < X, J_a Y >^2$, where $X, Y$ is an orthonormal basis of a horizontal 2-plane $\alpha \in \mathcal{H}_x$, $K(X, Y)$ denoting the sectional curvature of $\alpha$, and $K'(X', Y')$ denotes the sectional curvature in $M'$ of the 2-plane spanned by $X' = \pi_* X, Y' = \pi_* Y$.

**Theorem 8.** Let $M$ be an extrinsic hypersurface of a flat quaternionic Kähler manifold $\tilde{M}$ and let $\pi : M \to M'$ be a QR-submersion of $M$ on a quaternionic Kähler manifold $M'$. Then $M'$ is a quaternionic Kähler manifold with constant quaternionic sectional curvature $c > 0$, ($c = \|H\|$).

**References**


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