Variational calculus on sub-Riemannian manifolds

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Abstract

We provide invariant formulas for the Euler-Lagrange equation associated to sub-Riemannian geodesics. They use the concept of curvature and horizontal connection introduced and studied in the paper.

Key words: curvature, geodesics, connection

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1 Introduction

Consider a nonintegrable 2-dimensional distribution $x \rightarrow H_x$ in $\mathbb{R}^3 = \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}^t$ defined as $H = \ker \omega$, where $\omega$ is a 1-form on $\mathbb{R}^3$. The distribution $H$ is called the horizontal distribution. We shall assume the 1-form $\omega = \omega^1 dx_1 + \omega^2 dx_3 + \omega^3 dt$ has the coefficient $\omega^3 \neq 0$ so that dividing by it we may assume

$$\omega = -A_1(x) dx_1 + A_2(x) dx_2 + dt \quad (1.1)$$

with $A_1 = -\omega^1, and A_2 = \omega^2$. One may verify that

$$\omega(X_1) = \omega(X_1) = 0$$

where

$$X_1 = \partial_{x_1} + A_1(x) \partial_t , \quad X_2 = \partial_{x_2} - A_2(x) \partial_t \quad (1.2)$$

The vector fields $X_1, X_2$ span the horizontal distribution $H$ and they are called horizontal vector fields.

Suppose the 2-form

$$\Omega := d\omega = \left( \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_1} \right) dx_1 \wedge dx_2 \quad (1.3)$$

does not vanish. Then

$$[X_1, X_2] = -\left( \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_1} \right) \partial_t \notin \mathcal{H} \quad (1.4)$$

and then $\mathcal{H}$ is not integrable, by Frobenius theorem.

Consider the positive definite metric $g : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{F}$ in which the vector
fields \( \{X_1, X_2\} \) are orthonormal. The metric \( g \) is called the sub-Riemannian metric defined by the vector fields \( X_1 \) and \( X_2 \).

A curve \( s \to c(s) = (x_1(s), x_2(s), t(s)) \) is called horizontal curve if \( \dot{c}(s) \in \mathcal{H}_{c(s)} \), for every \( s \). As

\[
\dot{c}(s) = \dot{x}_1(s)\partial_{x_1} + \dot{x}_2(s)\partial_{x_2} + \dot{t}(s)\partial_t
\]

\[
= \dot{x}_1(s)X_1 + \dot{x}_2(s)X_2 + \omega(\dot{c}(s))\partial_t
\]

then \( c(s) \) is a horizontal curve iff

\[
\omega(\dot{c}) = \dot{t} - A_1(c)\dot{x}_1 + A_2(c)\dot{x}_2 = 0 \quad (1.5)
\]

The length of \( c \) with respect to the metric \( g \) is

\[
l(c) = \int_0^1 \sqrt{g(\dot{c}(s), \dot{c}(s))} \, ds = \int_0^1 \sqrt{\dot{x}_1(s)^2 + \dot{x}_2(s)^2} \, ds \quad (1.6)
\]

Given two points \( O \) and \( P \) there is at least a horizontal curve connecting them (see Chow [7]). The Carnot-Carathéodory distance is defined as

\[
d_C(O, P) = \inf\{l(c), c(0) = O, c(1) = P, c \text{ horizontal}\} \quad (1.7)
\]

The horizontal curve with minimum length are called sub-Riemannian geodesics and can be described using the Hamiltonian formalism as in the following (see Strichartz [11]).

Consider the sub-elliptic operator

\[
\Delta_X = \frac{1}{2}\left(X_1^2 + X_2^2\right) \quad (1.8)
\]

and define the Hamiltonian as the principal symbol of \( \Delta_X \)

\[
H(x, t, \xi, \theta) = \frac{1}{2}\left(\xi_1 + A_1(x)\theta\right)^2 + \frac{1}{2}\left(\xi_2 - A_2(x)\theta\right)^2 \quad (1.9)
\]

The projections on the \((x, t)\)-space of the solution of the Hamilton’s system

\[
\dot{x} = H_\xi, \quad \dot{t} = H_\theta \quad (1.10)
\]

\[
\dot{\xi} = -H_x, \quad \dot{\theta} = -H_t \quad (1.11)
\]

with the boundary conditions

\[
x(0) = t(0) = 0, \quad x(1) = x, t(1) = t \quad (1.12)
\]

are called sub-Riemannian geodesics between the origin and \((x, t)\).

From \( \dot{t} = H_\theta \) we get

\[
\dot{t} = A_1\dot{x}_1 - A_2\dot{x}_2 \quad (1.13)
\]

i.e. the sub-Riemannian geodesics are horizontal curves.
2 Connection and curvature

The horizontal connection
The horizontal connection is defined as

\[ D : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \] (2.14)

\[ D(V, W) = D_V W = \sum_{k=1,2} V g(W, X_k) X_k \] (2.15)

**Proposition 2.1** \( D \) is a linear metric connection.

*Proof:* One needs to verify the Leibnitz rule

\[ D_V (f W) = V(f) W + f D_V W \] (2.16)

and the condition

\[ U g(V, W) = g(D_U V, W) + g(V, D_U W) \] (2.17)

For the first part,

\[ D_V (f W) = \sum V g(f W, X_k) X_k \]

\[ = \sum V(f) g(W, X_k) X_k + f \sum V g(W, X_k) X_k \]

\[ = V(f) W + f D_V W \]

To show the second part,

\[ g(D_U V, W) + g(V, D_U W) = \]

\[ = g \left( \sum U g(V, X_i) X_i, W \right) + g \left( V, \sum U g(W, X_i) X_i \right) \]

\[ = g \left( \sum U (V^i) X_i, W \right) + g \left( V, \sum U (W^i) X_i \right) \]

\[ = \sum U (V^i) W^i + \sum U (W^i) V^i \]

\[ = U \left( \sum V^i W^i \right) = U g(V, W) \]

where \( V = \sum V^i X_i \) and \( W = \sum W^i X_i \).
Let $Z = Z^1X_1 + Z^2X_2$ be a horizontal vector field. The horizontal divergence is defined as

$$\text{div}_H Z = \text{trace}_g (V \rightarrow D_V Z) = \sum_k g(X_k, D_k Z) = \sum_k \left( X_k(Z^j)X_j \right)^k$$

The $X$-gradient of a function $f$ is defined as

$$\nabla_X f = \sum_k X_k(f) X_k$$

Then

$$\frac{1}{2} \text{div}_H \nabla_X = \Delta_X f$$

**The curvature tensor**

Let $\mathcal{K} : \mathcal{H} \rightarrow \mathcal{H}$ be defined as

$$\mathcal{K}(U) = \sum_k \Omega(U, X_k)X_k$$

$\mathcal{K}$ is $\mathcal{F}(\mathbb{R}^3)$-linear and can be considered as a $(1,1)$-tensor of curvature.

The following result shows that $\mathcal{K}$ is skew-selfadjoint.

**Proposition 2.2** For every $U, W \in \mathcal{H}$

$$g\left( \mathcal{K}(U), W \right) + g\left( U, \mathcal{K}(W) \right) = 0$$

**Proof:** We show first that

$$g\left( \mathcal{K}(U), W \right) = \Omega(U, W)$$

and using the skew-symmetry of $\Omega$ we get (2.22).

Indeed,

$$g\left( \mathcal{K}(U), W \right) = g\left( \sum_k \Omega(U, X_k)X_k, W \right) = \sum_k g(X_k, W)\Omega(U, X_k) = \Omega(U, W).$$
**Corollary 2.3** For any $U \in \mathcal{H}$,

$$g\left(\mathcal{K}(U), U\right) = 0. \quad (2.24)$$

The last result suggests that in the case of a 2-dimensional distribution, the curvature $\mathcal{K}$ is proportional with a rotation of angle $\pi/2$.

Define the rotation $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ as

$$\mathcal{J}(X_1) = X_2, \quad \mathcal{J}(X_2) = -X_1 \quad (2.25)$$

Then

$$\mathcal{K}(X_1) = \Omega(X_1, X_2)X_2 = \Omega(X_1, X_2)\mathcal{J}(X_1)$$
$$\mathcal{K}(X_2) = \Omega(X_2, X_1)X_1 = \Omega(X_1, X_2)\mathcal{J}(X_2)$$

We arrived at the following formula for the curvature

$$\mathcal{K}(U) = \Omega(X_1, X_2)\mathcal{J}(U), \quad \forall U \in \mathcal{H} \quad (2.26)$$

If the matrix $\Omega_{ij}$ is non-degenerate i.e. $\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_1}\right) \neq 0$, then $\mathcal{K}(U) \neq 0$ for $U \neq 0$.

If $V$ is not a horizontal vector field then the curvature can still be defined using

$$\mathcal{K}(V) = \sum_k \Omega(V, X_k)X_k \quad (2.27)$$

This is because the right hand side depends only on the horizontal part of $V$. Indeed, consider the vector field

$$V = V^1 \partial_{x_1} + V^2 \partial_{x_2} + V^3 \partial_t$$

A computation shows

$$V = \underbrace{V^1 X_1 + V^2 X_2}_{=V_H} + \omega(V) \partial_t$$

Then

$$\Omega(V, X_k) = \Omega(V_H, X_k) + \omega(V) \Omega(\partial_t, X_k) = 0$$

Hence $\mathcal{K}(V) = \mathcal{K}(V_H)$. 

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3 The Euler-Lagrange equation

The Legendre transform of the Hamiltonian (1.9) leads to the following Lagrangian

$$L(x, t, \dot{x}, \dot{t}) = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \theta \left( \dot{t} - A_1(x) \dot{x}_1 + A_2(x) \dot{x}_2 \right)$$  \hspace{1cm} (3.28)

where $\theta$ is constant because

$$\dot{\theta} = -\frac{\partial H}{\partial t} = -\frac{dH}{dt} = 0.$$  \hspace{1cm} (3.29)

We deal now with a minimization problem with constraints given by

$$L(c, \dot{c}) = \frac{1}{2} g(\dot{c}, \dot{c}) + \theta \omega(\dot{c})$$  \hspace{1cm} (3.30)

A computation shows the Euler-Lagrange system of equations

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{c}} = \frac{\partial L}{\partial c}, \quad c = (x_1, x_2, t)$$  \hspace{1cm} (3.31)

becomes

$$\ddot{x}_1 = \theta \left( \frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} \right) \dot{x}_2$$  \hspace{1cm} (3.32)

$$\ddot{x}_2 = -\theta \left( \frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} \right) \dot{x}_1$$  \hspace{1cm} (3.33)

If the velocity of the geodesic is given by $\dot{c}(s) = \dot{x}_1(s) X_1 + \dot{x}_2(s) X_2$, the system (3.32) – (3.33) can be written as

$$\ddot{x}_1 X_1 + \ddot{x}_2 X_2 = \theta \left( \frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} \right) (\dot{x}_2 X_1 - \dot{x}_1 X_2)$$  \hspace{1cm} (3.34)

The right hand side has the meaning of curvature. Indeed, using (2.25) and (2.26) the right hand side of (3.34) yields

$$-\theta \Omega(X_1, X_2) J(\dot{c}) = -\theta K(\dot{c}).$$  \hspace{1cm} (3.35)

For the left hand side of (3.34) consider the acceleration defined by the horizontal connection along $\dot{c}(s)$

$$D_{\dot{c}} \dot{c} = \sum_k \dot{c} g(\dot{c}, X_k) X_k = \dot{c}(\dot{x}_1) X_1 + \dot{c}(\dot{x}_2) X_2 = \ddot{x}_1 X_1 + \ddot{x}_2 X_2.$$  \hspace{1cm} (3.36)
Hence the Euler-Lagrange system of equations can be written globally as

\[ Dc \dot{c} = -\theta K(\dot{c}) \]  

(3.36)

In sub-Riemannian geometry the acceleration of the geodesics is equal to the curvature. This keeps the geodesics into the horizontal distribution. Like in Riemannian geometry, we have

**Corollary 3.1** The length of velocity \( \dot{c} \) in the sub-Riemannian metric \( g \) is constant.

*Proof:* As \( D \) is a metric connection,

\[ \dot{c} g(\dot{c}, \dot{c}) = 2g(Dc \dot{c}, \dot{c}) = -2\theta g(\mathcal{K}(\dot{c}), \dot{c}) = 0 \]

by Corollary 2.3.

**Hamilton-Jacobi equation**

**Lemma 3.2** Let \( c(s) = (x_1(s), x_2(s), t(s)) \) be a horizontal curve and a smooth function \( f \in \mathcal{F}(\mathbb{R}^3) \). Then

\[ \frac{df}{ds} = \frac{\partial f}{\partial s} + g(\dot{c}, \nabla_X f) \]  

(3.37)

*Proof:*

\[
\begin{align*}
\frac{df}{ds} &= \frac{\partial f}{\partial s} + \frac{\partial f}{\partial x_1} \dot{x}_1 + \frac{\partial f}{\partial x_2} \dot{x}_2 + \frac{\partial f}{\partial t} \dot{t} \\
&= \frac{\partial f}{\partial s} + \left( X_1 f - A_1(x) \frac{\partial f}{\partial t} \right) \dot{x}_1 \\
&\quad + \left( X_2 f + A_2(x) \frac{\partial f}{\partial t} \right) \dot{x}_2 + \frac{\partial f}{\partial t} \omega(\dot{c}) \\
&= \frac{\partial f}{\partial s} + (X_1 f) \dot{x}_1 + (X_2 f) \dot{x}_2 + \frac{\partial f}{\partial t} \omega(\dot{c}) \\
&= \frac{\partial f}{\partial s} + g(\dot{c}, \nabla_X f).
\end{align*}
\]

In the following we need to find the minimum of

\[ I = \int_0^\tau \frac{1}{2} (\dot{x}_1(s))^2 + (\dot{x}_2(s))^2 \, ds \]
= \int_0^\tau \frac{1}{2} |\dot{c}(s)|_g^2 \, ds

over the horizontal curves \( c(s) \) with fixed ends.

Let \( S(x, t) \in \mathcal{F} \) be the solution for the Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial \tau} + \frac{1}{2} |\nabla_X S|^2 = 0 \tag{3.38}
\]

\( S(O) = 0 \).

Consider the integral

\[
J = \int_0^\tau \frac{1}{2} |\dot{c}(s)|_g^2 \, ds - dS \tag{3.39}
\]

Using Lemma 3.2

\[
J = \int_0^\tau \left( \frac{1}{2} |\dot{c}(s)|_g^2 - \frac{\partial S}{\partial s} - g(\nabla_X S, \dot{c}) \right) \, ds
\]

\[
= \int_0^\tau \left( \frac{1}{2} |\dot{c} - \nabla_X S|_g^2 - \left( \frac{\partial S}{\partial s} + \frac{1}{2} |\nabla_X S|^2 \right) \right) \, ds
\]

\[
= \int_0^\tau \frac{1}{2} |\dot{c} - \nabla_X S|_g^2 \, ds \tag{3.40}
\]

The integrals \( I \) and \( J \) reach the minimum for the same horizontal curve and this occurs for a curve with the velocity

\[
\dot{c} = \nabla_X S \tag{3.41}
\]

**Theorem 3.3** A horizontal curve \( c(s) \) is energy-minimizing iff (3.41) holds.

Using (2.20) we get

**Corollary 3.4** The horizontal divergence of the geodesic flow is

\[
div_H \dot{c} = 2\Delta_X S \tag{3.42}
\]

**The Hamiltonian**
The Hamiltonian \( H : T^*M \to \mathbb{R} \) is defined as

\[
H(x, p) = \frac{1}{2} \sum_k p(X_k)^2
\]
If $p = df$, 
\[ H(x, df) = \frac{1}{2} \sum df(X_k)^2 = \frac{1}{2} \sum X_k(f)^2 = \frac{1}{2} |\nabla_X f|^2 \]

For $f = S$, 
\[ H(x, dS) = \frac{1}{2} |\nabla_X S|^2 = \frac{1}{2} |\dot{c}|^2 = \frac{1}{2} \]

We also have 
\[ H(x, \omega) = \frac{1}{2} \sum \omega(X_i)^2 = 0 \]

The eiconal equation
Consider the energy associated to a function $f \in \mathcal{F}(\mathbb{R}^3)$ defined as 
\[ H(\nabla f) = H(x, df) = \frac{1}{2} |\nabla_X f|^2 = \frac{1}{2} \left( (X_1 f)^2 + (X_2 f)^2 \right) \]
We also have 
\[ H(x, \omega) = \frac{1}{2} \sum \omega(X_i)^2 = 0 \]

The eiconal equation
Consider the energy associated to a function $f \in \mathcal{F}(\mathbb{R}^3)$ defined as 
\[ H(\nabla f) = H(x, df) = \frac{1}{2} |\nabla_X f|^2 = \frac{1}{2} \left( (X_1 f)^2 + (X_2 f)^2 \right) \]

The front wave is given by the level curves of the energy and it is described by the eiconal equation 
\[ H(\nabla f) = k, \quad \text{positive constant} \]
with the initial condition 
\[ f(O) = 0 \]

If $k = 0$, then $f$ is the constant function equal to zero. Indeed, suppose that $f$ is not constant. There is a point $p$ such that $(\text{grad} f)_p \neq 0$. Then $\Sigma_c = f^{-1}(c)$ will be a surface through $p$, where $c = f(p)$. As $X_i(f) = 0$, then $X_i$ are tangent to $\Sigma_c$ on a neighborhood of $p$ and hence $\Sigma_c$ becomes integral surface for the horizontal distribution $\mathcal{H}$ around $p$, which is nonintegrable, contradiction.

If $k \neq 0$, consider the geodesics starting at origin $c(0) = O$ parametrized such that $|\dot{c}(s)|^2_g = 2k$. If $S$ is the action along $c(s)$, by (3.41) 
\[ H(\nabla S) = \frac{1}{2} |\nabla_X S|^2_g = \frac{1}{2} |\dot{c}|^2_g = k \]

Jacobi vector fields and curvature
Let $c(s)$ be a subRiemannian geodesic which starts at origin and let $P$ be the first conjugate point with 0 along $c(s)$. Denote by $V(s)$ a Jacobi vector field along $c(s)$ and by $S(s)$ the action between 0 and $c(s)$. 

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Proposition 3.5
\[
\int_0^1 K(V(s))(S(s)) \, ds = 0 \tag{3.46}
\]

where \( P = c(1) \) and \( K \) is the curvature.

Proof:
Let \( c_\epsilon = F_\epsilon(c) \) be a smooth variation of \( c \), such that for every \( \epsilon \), \( c_\epsilon \) is a subRiemannian geodesic. As \( c_\epsilon \) is a horizontal curve, then
\[
0 = \int_0^1 \omega(\dot{c}_\epsilon(s)) \, ds = \int_{c_\epsilon} \omega = \int_{F_\epsilon(c)} F_\epsilon^* \omega = \int_c F_\epsilon^* \omega
\]
Then
\[
\frac{d}{d\epsilon} \int_c F_\epsilon^* \omega = 0
\]
or,
\[
\int_c L_V \omega = 0
\]
where \( V \) is the Jacobi vector field associated to the variation \( (c_\epsilon)_\epsilon \).

As \( V \) is zero at the end points of \( c \),
\[
\int_C d(i_V \omega) = \int_{\partial C} i_V \omega = \omega(V)(0) - \omega(V)(1) = 0
\]
Using Cartan decomposition
\[
L_V \omega = d(i_V \omega) + i_V (d\omega)
\]
we get
\[
\int_c i_V \Omega = 0
\]
which can also be written as
\[
\int_0^1 \Omega(V(s), \dot{c}(s)) \, ds = 0
\]
Using \( \dot{c} = \sum \dot{c}^j X_j \) and \( \dot{c}^j(s) = X_j(S) \), then
\[
\Omega(V, \dot{c}) = \Omega(V, \dot{c}^j X_j) = \dot{c}^j \Omega(V, X_j) = \Omega(V, X_j) X_j(S) = K(V)(S).
\]
Hence
\[
\int_0^1 K(V)(S) \, ds = 0
\]
4 Constant curvature flow

In this section we ask the problem to a vector field such that $|K(V)|^2 = 1$. As a nondegenerate, closed 2-form, $\Omega$ can be regarded as a symplectic form. One may associate the horizontal Hamiltonian vector field $X_f$ to a function $f$ as

$$\Omega(X_f, W) = W(f), \quad \forall W \in \mathcal{H} \quad (4.47)$$

We shall show that $X_f$ has constant curvature for a certain $f$.

$$K(X_f) = \sum \Omega(X_f, X_k)X_k = \sum X_k(f)X_k = \nabla_X f$$

and choosing $f = S$

$$|K(X_S)|^2_g = |\dot{c}|^2_g = 1 \quad (4.48)$$

In the following we find the relation between the Hamiltonian field $X_S$ and the geodesic flow $\dot{c}$.

Applying (2.26),

$$K(K(U)) = \Omega(X_1, X_2)K(J(U))$$

$$= \Omega(X_1, X_2)^2J^2(U) = -\Omega(X_1, X_2)^2U$$

or

$$K^2 = -\Omega(X_1, X_2)^2I d \quad (4.49)$$

Using (4.49)

$$X_S = -\Omega(X_1, X_2)^{-2}K^2(X_S)$$

$$= -\Omega(X_1, X_2)^{-2}K(\dot{c}) = -\Omega(X_1, X_2)^{-1}J(\dot{c})$$

or

$$\dot{c} = \Omega(X_1, X_2)J(X_S) \quad (4.50)$$

5 A few examples of sub-Riemannian manifolds

5.1 The Heisenberg group $\mathbb{H}_1$

The Heisenberg group constitutes the paradigm of the theory. The 3-dimensional Heisenberg group can be realized as $\mathbb{H}_1 = \mathbb{R}^3 \times \mathbb{R} = \{(x, t)\}$ endowed with the group law

$$(x, t) * (x', y') = (x + x', t + t' + 2x_2x_1' - 2x_1x_2') \quad (5.51)$$
The vector fields which generate the nonintegrable distribution $\mathcal{H}$ are

$$
X_1 = \partial_{x_1} + 2x_2 \partial_t, \quad X_2 = \partial_{x_2} - 2x_1 \partial_t, \quad T = \partial_t
$$

They are left invariant with respect to the group law and generate the Lie algebra of $\mathbb{IH}_1$. The 1-form is

$$
\omega = dt - 2x_2 \dot{x}_1 + 2x_1 \dot{x}_2
$$

and the curvature 2-form is

$$
\Omega = 4 dx_1 \wedge dx_2
$$

and

$$
\Omega(X_1, X_2) = 4
$$

and the curvature given by (2.26) becomes

$$
\mathcal{K}(U) = 4\mathcal{J}(U), \quad \forall U \in \mathcal{H}
$$

The Euler-Lagrange equation is

$$
\ddot{x} = 4\theta \mathcal{J}(\dot{x}).
$$

### 5.2 The $(2n+1)$-dimensional Heisenberg group $\mathbb{IH}_n$

The $2n$-vector fields are defined on $\mathbb{R}^{2n+1}$ as

$$
X_k = \partial_{x_k} + B_k(x) \partial_t, \quad k = 1, 2, \ldots, 2n
$$

where

$$
B_j(x) = \sum_{k=1}^{2n} 2a_{jk} x_k
$$

or $B = 2Ax$ where $A$ is a skew-symmetric non-singular matrix. The 1-form of connection in this case is

$$
\omega = dt - 2Ax dx
$$

Then the 2-form becomes

$$
\Omega = d\omega = 2 \sum_{p,j=1}^{2n} a_{pj} dx_p \wedge dx_j = -2\langle A dx, dx \rangle
$$
A computation shows that the curvature along a horizontal vector field $U$ is

$$K(U) = - \sum_{k,p=1}^{2n} 4a_{pk} U^k X_p$$

(5.62)

The Euler-Lagrange equation system of equations is given by

$$\ddot{x} = -4\theta K(\dot{x})$$

(5.63)

### 5.3 A step 4 case

Consider the vector fields

$$X_1 = \partial_{x_1} + 4x_2|x|^2 \partial_t, \quad X_2 = \partial_{x_2} - 4x_1|x|^2 \partial_t$$

(5.64)

which define the 1-form

$$\omega = dt - 4|x|^2(x_2 dx_1 - x_1 dx_2)$$

(5.65)

Then

$$\Omega = 16|x|^2 dx_1 \wedge dx_2$$

(5.66)

The curvature becomes

$$K(U) = 16|x|^2 J(U), \quad \forall U \in \mathcal{H}$$

(5.67)

The Euler-Lagrange system is

$$\ddot{x} = 16\theta|x|^2 J(\dot{x})$$

(5.68)

### 5.4 A step 3 case

The vector fields

$$X_1 = \partial_{x_1} + \frac{x_2}{2} \partial_t, \quad X_2 = \partial_{x_2}$$

(5.69)

define the Martinet distribution on $\mathbb{R}^3$. Then

$$\omega = dt - \frac{1}{2} x_2^2 dx_1$$

(5.70)

and

$$\Omega = x_2 dx_1 \wedge dx_2$$

(5.71)

The curvature is

$$K(U) = x_2 J(U)$$

(5.72)
References


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