INVARIANT SIGNATURES OF CLOSED PLANAR CURVES

EMILIO MUSSO AND LORENZO NICOLODI

In memory of Professor Aristide Sanini

ABSTRACT. We prove that any subset of $\mathbb{R}^2$ parametrized by a C^1 periodic function and its derivative is the Euclidean invariant signature of a closed planar curve. This solves a problem posed by Calabi *et al.* in [6]. Based on the proof of this result, we then develop some cautionary examples concerning the application of signature curves for object recognition and symmetry detection as proposed in [6].

1. INTRODUCTION

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ be a unit-speed curve defined on some open interval I. The corresponding *Euclidean invariant signature* is the set $\mathcal{S} \subset \mathbb{R}^2$ parametrized by $(\kappa, \dot{\kappa})$, where κ is the curvature of γ and $\dot{\kappa}$ is its derivative. Recently, this notion has received considerable attention in computer vision, mostly in the issues of object recognition and symmetry detection (cf. [5], [3], [4], [6], [1], [17], [13], and the literature therein). It has also played a relevant role in the study of invariant variational problems of differential geometry and mathematical physics (cf. [12], [14], [11], [15], [16], [9]). In [6], Section 5, Calabi *et al.* indicated as a fundamental open problem the characterization of those subsets of $\mathbb{R}^2$ which are the signatures of closed curves. It is clear that the signature of a closed curve is a closed *phase portrait*, that is, a subset of $\mathbb{R}^2$ parametrized by a periodic function of class at least C^1 and its derivative (cf. [12]). The main purpose of this paper is to prove the following result.

Theorem 1. *Any closed phase portrait which does not degenerate into a single point is the Euclidean signature of a 1-parameter family of non congruent unit-speed closed curves of class at least C^3.*

The characterization of those phase portraits which correspond to simple closed curves still remains an open problem.

The idea of using the signature in computer vision relies on the paradigm that the signature fully determines the shape of the curve and, via the index of the signature map,[1] the order of its symmetry group (cf. [6], [1], [17], [18], [19]). A

Date: Version of April 20, 2009.

2000 *Mathematics Subject Classification.* 53A04; 65D17; 65D18.

Key words and phrases. Closed curves; Signature; Phase portrait; Symmetry; Object recognition; Symmetry detection.

Authors partially supported by MIUR projects: *Metriche riemanniane e varietà differenziabili* (E.M.); *Proprietà geometriche delle varietà reali e complesse* (L.N.); and by the GNSAGA of INDAM.

[1]See Section 2 for the definition.

thorough analysis of Theorem 1 points out possible difficulties in the application of the signature method if interpreted too broadly. In fact, based on the proof of Theorem 1, we provide a general procedure to construct families of closed curves of class C^3, not necessarily simple, with the same signature, but which are not congruent to each other because of different length. Moreover, numerical experiments show that a non-convex simple closed curve with non-trivial symmetry group possesses a continuous deformation through simple, non-convex, closed curves with the same signature but different length. A natural question is whether, in the smooth category, there exist isosigned deformations of non-congruent, convex, simple closed curves with equal length. We answer this question in Section 4, where a detailed study of simple closed curves with symmetries leads to the following result.

Theorem 2. *There exist 1-parameter families* $\{\Gamma_r\}_{r\in[0,\eta)}$ *of smooth strictly convex simple closed curves which have the same signature and are not congruent. Moreover, these families can be constructed so that* Γ_0 *is not locally congruent to* Γ_r*, for each* $r \in (0,\eta)$.

The isosigned curves of a deformation $\{\Gamma_r\}$ all have the same length, symmetry group and index. By further specializing the proof of Theorem 2, it is also possible to construct deformations whose isosigned curves all have the same length and symmetry group, but different indices. A problem that remains open is to characterize, within the class of simple closed curves without symmetries, those curves whose shape is uniquely determined by the signature. In Section 5, we deal with the question of symmetry detection and prove the following theorem.

Theorem 3. *Let* G_1, $G_2 \subset \mathrm{SO}(2)$ *be two finite subgroups of order* q_1 *and* q_2*, respectively. Then, there exist smooth simple closed curves* Γ_1 *and* Γ_2 *with the same signatures, the same indices, and with symmetry groups* G_1 *and* G_2*, respectively. In particular, if* $G_1 \neq G_2$*, the curves cannot be congruent to each other.*

The proof is based on the construction of a special family of smooth simple closed curves (the "cogwheels") which have identical signatures, but are not congruent to each other. Yet the curves of this family are locally congruent by construction.[2]

The above results tell that some caution is needed in the application of signature curves to object recognition and symmetry detection as proposed by Calabi *et al.* [6]. Actually, that the signature uniquely determines the original curve up to a rigid motion is certainly true in the real-analytic category, in the sense that two real-analytic curves with the same signature are congruent to each other. It is also true for (non closed) curves of class at least C^4 and with no vertices. More generally, the possibility of reconstructing the curve from its signature requires that the signature have constant rank along the curve (cf. the regularity hypothesis of Theorem 14.9 in [8] and Theorem 5.2 in [6]). Observe that our counterexamples do not satisfy this regularity assumption. As suggested by one of the referees, it is likely that the signature would uniquely determine the shape of a smooth closed curve under additional information on the connected components of the vertex set (number and measure of the components) and the main global invariants (order of

[2]We recall that two curves Γ_1 and Γ_2 are *locally congruent* if, for each $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$, there exist open neighborhoods $\mathcal{U}_1,\mathcal{U}_2 \subset \mathbb{R}^2$ of p_1 and p_2, respectively, and $F_1 \in \mathbb{E}(2)$ such that $F_1(\Gamma_1 \cap \mathcal{U}_1) = \Gamma_2 \cap \mathcal{U}_2$, and conversely, for each $p_2 \in \Gamma_2$ and $p_1 \in \Gamma_1$, there exist open neighborhoods $\mathcal{U}_1,\mathcal{U}_2 \subset \mathbb{R}^2$ of p_1 and p_2, respectively, and $F_2 \in \mathbb{E}(2)$ such that $F_2(\Gamma_2 \cap \mathcal{U}_2) = \Gamma_1 \cap \mathcal{U}_1$.

the symmetry group, length and turning number). Similar considerations can be made for the question of symmetry detection. In particular, if the curve is analytic, the symmetry result stated in [6] holds true.

The paper is organized as follows. Section 2 recalls the basic definitions and collects some preliminary results. Section 3 proves Theorem 1, develops the general construction of isosigned deformations and discusses some numerical experiments. Section 4 proves Theorem 2. Section 5 first discusses a special class of simple closed curves, the "cogwheels", and then proves Theorem 3.

Acknowledgments. We would like to thank the referees for their useful comments and suggestions.

2. Preliminaries

In this section we collect some preliminary results about plane curves and recall the basic definitions.

2.1. **Preliminary results.** Let $\gamma : [0, L] \to \mathbb{R}^2$ be a parametrization by arclength of an immersed curve $\Gamma \subset \mathbb{R}^2$ of class C^h, $h \geq 3$. Then, for each $s \in [0, L]$, $(\dot{\gamma}(s), J\dot{\gamma}(s))$ is a positively oriented orthonormal basis of $\mathbb{R}^2$ with the usual orientation. Here the dot denotes the derivative with respect to the arc element and J the counterclockwise rotation by $\pi/2$ in the plane. For simplicity, we also require that $\gamma(0)$ coincides with the origin of $\mathbb{R}^2$, so that the arclength parametrization is completely determined. The unit tangent vector $\dot{\gamma}(s)$ satisfies the Frenet equations

$$\ddot{\gamma}(s) = \kappa(s) J\dot{\gamma}(s),$$

where $\kappa(s) = \langle \ddot{\gamma}(s), J\dot{\gamma}(s) \rangle$ is the (signed) curvature function. If the function κ is given in advance, the Frenet equations together with the initial conditions $\gamma(0) = (0, 0)$ and $\dot{\gamma}(0) = e_1 = (1, 0)$, can be solved explicitly. The arclength parametrization of the associated curve Γ is given by

$$\gamma(s) = \int_0^s e^{i\theta(u)} du,$$

where

$$\theta(s) = \int_0^s \kappa(u) du$$

is the *angle of inclination.* Any other curve Γ^* with curvature κ as a function of the arclength is congruent to Γ, i.e. there exists a rigid motion of the plane, $A \in \mathbb{E}(2)$, such that $\Gamma^* = A(\Gamma)$.

Let $\Gamma \subset \mathbb{R}^2$ be a closed non-circular immersed curve of class C^h, $h \geq 3$. Then Γ admits an arclength parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$ that is periodic of minimal period L, being L the length of the curve. If, in addition, γ is injective on $[0, L)$, then Γ is a *simple* closed curve, that is, Γ is embedded and diffeomorphic to S^1. If Γ is a closed curve, then the following conditions hold:

(1) $\kappa(s)$ is periodic function of class C^{h-2}, with minimal period ℓ dividing L;
(2) the *total curvature* $\int_0^L \kappa(u)du$ is an integral multiple of 2π;
(3) The vector $\int_0^L e^{i\theta(u)}du$ vanishes identically.

Remark 1. In fact, conditions (1), (2) and (3) are also sufficient for closedness (cf. the proof of Lemma 4, or [10]).

The minimal period ℓ is called the *reduced length* of the curve, while the positive integer q such that $L = q\ell$ is referred to as the *index of symmetry* of the curve. If $q = 1$, the curve is said *irreducible*; otherwise, *reducible.* The integer p such that

$$\int_0^L \kappa(u)du = 2\pi p$$

is the *turning number* of the curve. It is well-known that for a positively-oriented simple closed curve the turning number $p = 1$.

We will need the following technical lemma.[3]

Lemma 4. *Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a periodic function of class C^{h-2}, $h \geq 3$, with minimal period ℓ, and assume that*

$$\frac{1}{2\pi}\int_0^\ell \kappa(u)du = \frac{p}{q} \in \mathbb{Q} \setminus \mathbb{Z}, \tag{2.1}$$

where $q > 1$ and p are two relatively prime integers. Then, the corresponding unit-speed curve γ is closed of length $L = q\ell$.

Proof. Since the problem is invariant under dilations, the discussion can be reduced to the case where κ has minimal period $\ell = 2\pi$ and Fourier expansion

$$\kappa(s) = \frac{p}{q} + \sum_{n=1}^{\infty} \left(a_n \cos ns + b_n \sin ns\right). \tag{2.2}$$

If we set

$$\theta(s) = \frac{p}{q}s + \sum_{n=1}^{\infty} \left(\frac{a_n}{n}\sin ns - \frac{b_n}{n}\cos ns\right), \tag{2.3}$$

then the curve defined by κ is given by

$$\gamma(s) = \int_0^s e^{i\theta(u)}du. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\gamma(s + 2\pi) = e^{2\pi i\frac{p}{q}}\gamma(s) + \mathcal{E}, \tag{2.5}$$

where

$$\mathcal{E} = \int_0^\ell e^{i\theta(u)}du.$$

Proceeding inductively, we have

$$\gamma(s + 2m\pi) = e^{2\pi i\frac{p}{q}m}\left(\gamma(s) + \mathcal{E}\sum_{h=1}^{m} e^{-2i\pi h\frac{p}{q}}\right), \quad m \in \mathbb{Z}. \tag{2.6}$$

Substitution of

$$\sum_{h=1}^{m} e^{-2i\pi\frac{p}{q}} = e^{-i(m+1)\pi\frac{p}{q}}\csc\left(\frac{\pi p}{q}\right)\sin\left(\frac{m\pi p}{q}\right), \quad m \in \mathbb{Z}, \tag{2.7}$$

into (2.6) yields

$$\gamma(s + 2m\pi) = e^{2im\pi\frac{p}{q}}\left(\gamma(s) - C\right) + C, \quad m \in \mathbb{Z}, \tag{2.8}$$

[3]Though elementary, we could not find a proof of this result in the literature. Some related material is given in [7], [2].

where
$$C = \frac{i}{2}\csc\left(\frac{\pi p}{q}\right) e^{-i\pi\frac{p}{q}}\mathcal{E}.$$
This completes the proof of the lemma. □

Remark 2. The curve γ has turning number p and center of symmetry C.

More specifically, we can get sufficient conditions for a curve with strictly positive periodic curvature being closed and simple. This will be used in the proof of Theorem 2.

Corollary 5. *Let $\kappa : \mathbb{R} \to \mathbb{R}$ be a smooth strictly positive periodic function with minimal period $\ell = 2\pi$ such that*

$$\frac{1}{2\pi}\int_0^{2\pi} k(u)du = \frac{1}{q}, \tag{2.9}$$

where $q > 1$ is an integer. Then the corresponding unit-speed curve γ is closed and simple of length $L = 2q\pi$.

Proof. It follows from Lemma 4 that
$$\gamma(s + 2q\pi) = \gamma(s),$$
which implies that γ is closed of length $L = 2q\pi$ and with turning number
$$p = \frac{\theta(L) - \theta(0)}{2\pi} = \frac{1}{2\pi}\int_0^L \kappa(u)du = 1.$$
Suppose now that γ is not simple. Then, after possibly a change of the form $s \to s + a$, we may assume that there exists $s_1 \in (0, L)$ such that $\gamma(0) = \gamma(s_1)$. We now get a contradiction, and hence the proof of the lemma, by proving that
$$\frac{1}{2\pi}\int_0^{s_1} \kappa(u)du > \frac{1}{2}, \quad \frac{1}{2\pi}\int_{s_1}^L \kappa(u)du > \frac{1}{2}.$$
Up to the action by a rigid motion, we may assume that
$$\gamma(0) = (0, 0), \quad \gamma'(0) = (1, 0).$$
Then, if $\gamma(s) = (x(s), y(s))$, the smooth function
$$s \in [0, s_1] \mapsto y(s) \in \mathbb{R}$$
is such that $y(0) = y(s_1) = 0$ and $y(s) > 0$ for some $s \in (0, s_1)$. Then it attains a maximum at some $s_2 \in (0, s_1)$. Thus, $y'(s_2) = 0$, i.e. $\gamma'(s_2) = \pm(1, 0)$. Since κ is strictly positive, the angular function θ is strictly increasing, which implies $\gamma'(s_2) = (1, 0)$, that is $\theta(s_2) = \pi$. This yields the first inequality. The second inequality follows by the same argument. □

Remark 3. By a symmetry of the parametrized immersed curve $\gamma : \mathbb{R} \to \Gamma \subset \mathbb{R}^2$ it is meant an element $A \in \mathbb{E}(2)$ such that $A(\gamma(t)) = \gamma(t + \ell_A)$, $\ell_A \in \mathbb{R}$. The group $G(\gamma)$ of all such symmetries is a subgroup of $G(\Gamma)$, the group consisting of all $A \in \mathbb{E}(2)$ such that $A(\Gamma) = \Gamma$.

If Γ is a closed curve with index of symmetry $q > 1$, the integer q is the order of the group $G(\gamma)$ that consists of all rotations by multiples of the angle $\omega = 2\pi p/q$ about a fixed point C. The subset $\Gamma_0 = \gamma([0, \ell)) \subset \Gamma$ is the *fundamental domain* of Γ with respect to the action of $G(\gamma)$. If the curve is irreducible, then $\Gamma_0 = \Gamma$; otherwise, Γ is obtained from the fundamental domain via $G(\gamma)$,

$$\Gamma = \bigcup_{A \in G(\gamma)} A(\Gamma_0).$$

If Γ is a simple closed curve, the group $G(\gamma)$ coincides with $G(\Gamma)$, and the set of its symmetries is a finite subgroup of rotations around a point. Its cardinality is the order of symmetry of the curve.

2.2. **Signature and index.**

Definition 1. Let $\gamma : t \in \mathbb{R} \to \gamma(t) \in \mathbb{R}^2$ be a parametrized immersed curve of class C^h. The *signature map* associated to γ is the mapping of class C^{h-3} defined by

$$\sigma_\gamma : t \in \mathbb{R} \mapsto \left(\kappa(t), \frac{1}{\|\gamma'(t)\|}\kappa'(t)\right) \in \mathbb{R}^2,$$

where the prime denotes the derivative with respect to the parameter t and

$$\kappa(t) = \langle \gamma''(t), J\gamma'(t)\rangle / \|\gamma'(t)\|^3$$

is the curvature function of the parametrized curve γ. The image of σ_γ will be denoted by $\mathcal{S}(\gamma)$ and referred to as *signature of the parametrized curve* γ.

Definition 2. The *index* of the signature map σ_γ is given by

$$\operatorname{ind}\sigma_\gamma = \min\left\{\#\sigma_\gamma^{-1}(\zeta) \,|\, \zeta \in \mathcal{S}(\gamma)\right\}.$$

For a simple closed curve Γ, the set $\mathcal{S}(\gamma)$ is independent of the parametrization γ of Γ. This set is referred to as the *Euclidean signature curve* associated with Γ. It is characterized by its parametrization in terms of $\kappa(s)$ and $\dot\kappa(s)$ with respect to the arc element:

$$\mathcal{S}(\Gamma) := \{(\kappa(s), \dot\kappa(s)) \,:\, s \in \mathbb{R}\} \subset \mathbb{R}^2.$$

Remark 4. If Γ is closed but not simple, then $\mathcal{S}(\gamma)$ depends on the equivalence class of the parametrization γ of Γ. For example, the curve drawn in Figure 1 admits non-equivalent parametrizations with different signatures. Note that this curve has non-transversal self-intersections. It is likely that one can canonically associate a signature curve to any closed curve with transversal self-intersections. Though, we have not a proof of this fact.

Remark 5. The signature curve of a simple closed curve is invariant under the action of the Euclidean group, that is, if Γ and Γ' are two congruent simple closed curves, then the corresponding signatures coincide.

Remark 6. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be two smooth functions. We say that f_1 is a *first-order deformation* of f_2 if for each $t_1 \in \mathbb{R}$ there exists $t_2 \in \mathbb{R}$ such that $f_1(t_1) = f_2(t_2)$ and $f'(t_1) = f_2'(t_2)$. Observe that the unit-speed parametrized curves corresponding to curvature functions which are first-order deformations of each other have identical signatures.

Remark 7. The signature of a closed unit-speed parametrized curve of class at least C^3 is a subset of $\mathbb{R}^2$ parametrized by a periodic function of class at least C^1 and its derivative, a closed phase portrait.

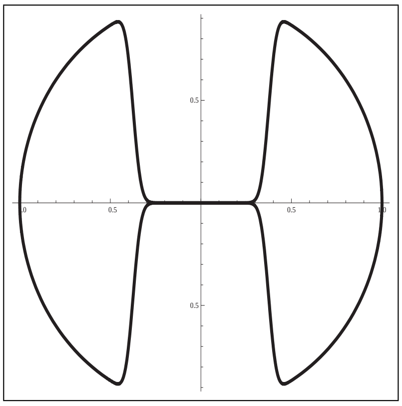

FIGURE 1

FIGURE 2. A curve admitting non-equivalent parametrizations with different signatures.

Finally, if $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a C^h parametrized immersion, we recall that $\gamma(t_0)$ is a *vertex* of γ if t_0 is a critical point of the curvature, i.e. $\kappa'(t_0) = 0$, $t_0 \in \mathbb{R}$. If $\kappa(t) = \text{const}$, for $t_1 \leq t \leq t_2$, all the $\gamma(t)$ are vertices. If $\gamma(t_0)$ is a vertex, then

$$\sigma_\gamma(t_0) \in \mathcal{S}(\gamma) \cap \{x\text{-axis}\}.$$

Moreover, $\gamma(t_0)$ is called an *inflection point* of γ if $\kappa(t_0) = 0$. If $\gamma(t_0)$ is an inflexion point, then

$$\sigma_\gamma(t_0) \in \mathcal{S}(\gamma) \cap \{y\text{-axis}\}.$$

3. Theorem 1: Proof and consequences

3.1. **Proof of Theorem 1.** Let $\mathcal{S} \subset \mathbb{R}^2$ be a closed phase portrait parametrized by $(\kappa, \dot{\kappa})$, where κ is a non-constant periodic function of class C^1 with minimal period ℓ. Using the invariance under dilations we may assume $\ell = 2\pi$. Possibly translating the independent variable, we may also assume that $\kappa(s)$ attains its maximum κ_0 at $s = 0$ and its minimum κ_1 at $s_1 \in (0, 2\pi)$. Set

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \kappa(u) du$$

and choose a positive constant m and two relatively prime integers $q > 1$ and p, such that the line Λ of equation

$$\kappa_1 m x - \kappa_0 y + 2\pi \left(\frac{p}{q} - a_0 \right) = 0$$

intersects the positive quadrant

$$\mathcal{Q} = \left\{ (x, y) \in \mathbb{R}^2 \,|\, x > 0,\, y > 0 \right\}.$$

For instance, it is enough to take p and q such that

$$\frac{2\pi}{\kappa_1} \left(\frac{p}{q} - a_0 \right) > 0.$$

Define

$$\begin{aligned}\delta_1(r) &= mr,\\ \delta_0(r) &= \frac{2\pi}{\kappa_0}\left(\frac{p}{q}-a_0\right)-\frac{\kappa_1}{\kappa_0}mr.\end{aligned}$$

Since $\Lambda \cap \mathcal{Q} \neq \emptyset$, δ_0 is positive on a closed interval $I = [0, \eta] \subset \mathbb{R}^+$, $\eta > 0$. Next, for each $r \in I$, let $K(-, r)$ be the C^1 periodic function, with minimal period

$$\ell_r = 2\pi + \delta_0(r) + \delta_1(r),$$

defined by

$$K(s,r) = \begin{cases} \kappa_0, & s \in [0, \delta_0(r)],\\ \kappa(s-\delta_0(r)), & s \in [\delta_0(r), s_1+\delta_0(r)],\\ \kappa_1, & s \in [s_1+\delta_0(r), s_1+\delta_0(r)+\delta_1(r)],\\ \kappa(s-\delta_0(r)-\delta_1(r)), & s \in [s_1+\delta_0(r)+\delta_1(r), \ell_r].\end{cases}$$

From the definition of $K(-, r)$, we have

$$\frac{1}{2\pi}\int_0^{\ell_r} K(u,r)du = \frac{p}{q}.$$

By Lemma 4, the corresponding unit-speed curve γ_r is of class C^3 and closed, but non necessarily simple. Moreover, since γ_r has length $L_r = q\ell_r$, the curves of the family $\{\gamma_r\}_{r\in I}$ are not congruent to each other. By definition, all the signature maps

$$\sigma_r : s \in \mathbb{R} \mapsto \sigma_r(s) = (K(s,r), \partial_s K(s,r)) \to \mathbb{R}^2,$$

have the given phase portrait $\mathcal{S}$ as their signature. This concludes the proof.

3.2. **Consequences: An explicit construction.** Let γ be a unit-speed closed planar curve. The proof of Lemma 4 indicates an explicit method to construct a 1-parameter family of closed curves $\{\gamma_r\}_{r\in I}$ which have the same signature curve of γ but are not congruent to each other. The construction can be summarized as follows:

- Take a C^n periodic function f of period 2π, with $n \geq 1$ satisfying

$$\int_0^{2\pi} f(u)du = 0$$

  and

$$\begin{cases} m_0 = \max\{f(s) : s \in [0, 2\pi]\} > 0,\\ m_1 = \min\{f(s) : s \in [0, 2\pi]\} < 0.\end{cases}$$

- Find $s_0, s_1 \in [0, 2\pi)$ such that $f(s_0) = m_0$ and $f(s_1) = m_1$. Without loss of generality we may assume $s_0 < s_1$.
- For each $r \geq 0$, put

$$\delta_0(r) = -m_1 \cdot r, \quad \delta_1(r) = m_0 \cdot r, \quad \ell_r = \delta_0(r) + \delta_1(r) + 2\pi.$$

  For every $a, b \in \mathbb{R}$, $a < b$, denote by $B(-|a, b) : \mathbb{R} \to \mathbb{R}$ the unit-step function

$$B(s|a,b) = \begin{cases} 1, & s \in [a,b],\\ 0, & s \notin [a,b].\end{cases}$$

For each $r > 0$, consider the functions $f_j(-, r) : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, 5$, given by

$$\begin{cases} f_1(s,r) = B(s|0, s_0)f(s), \\ f_2(s,r) = B(s|s_0, \delta_0(r) + s_0)m_0, \\ f_3(s,r) = B(s|s_0 + \delta_0(r), s_1 + \delta_0(r))f(s), \\ f_4(s,r) = B(s|s_1 + \delta_0(r), s_1 + \delta_0(r) + \delta_1(r))m_1, \\ f_5(s,r) = B(s|s_1 + \delta_0(r) + \delta_1(r), \ell_r)f(s). \end{cases}$$

Set

$$F_0(s,r) = \sum_{i=1}^{5} f_i(s,r).$$

This is a piecewise C^n function such that $\operatorname{supp}(F_0(-,r)) \subset [0, \ell_r]$. Next, define

$$F(s,r) = \sum_{j\in\mathbb{Z}} F_0(s - j \cdot \ell_r, r).$$

The function $F(-,r) : \mathbb{R} \to \mathbb{R}$ is of class at least C^1 and periodic of period ℓ_r. If f is smooth and s_0, s_1 are critical points of order $h \geq 1$ (i.e. all the derivatives $f^{(i)}$ vanish at s_0 and s_1, for every $i \leq h$), then $F(-,r)$ is of class C^h. In particular, one can construct smooth (but obviously non real-analytic) examples.

- Choose two relatively prime integers p, q, with $q > 1$ and set

$$K(s,r) = F(s,r) + \frac{2\pi \cdot p}{\ell_r \cdot q}.$$

Then $K(-,r)$ is a periodic function of period ℓ_r such that

$$\frac{1}{2\pi}\int_0^{\ell_r} K(u,r)du = \frac{p}{q}.$$

- From Lemma 2, it follows that the unit-speed curve γ_r defined by the curvature function $K(-,r)$ is closed and of class at least C^3. All the curves of this family have the same signature but are not congruent to each other.
- Solve the Frenet system

$$\dot{x} = y, \quad \dot{y} = K(s,r)z, \quad \dot{z} = -K(s,r)y$$

with standard numerical routines (in most cases it is enough to use the Runge–Kutta method).
- Compute the numerical approximations of γ_r and proceed with the visualization.

3.3. **Numerical examples and experimental evidences.** We now analyze two examples and show how to implement standard numerical routines in our general scheme. The programs for the numerical computations and for the visualization have been performed with the software MATHEMATICA 6. For completeness, we add the main steps of the program used for the computations and the visualization.

Example 1. Start with the periodic function

$$f(s) = \frac{1}{2}(\sin(s) - \cos(3s))$$

Step I. Define f and compute m_0, m_1, s_0 and s_1:

```
f[s_]:=(1/2)(Sin[s] − Cos[3s]);
NMX:=NMaximize[{f[s], 0 ≤ s < 2Pi}, {s}, Method → "DifferentialEvolution"];
NMN:=NMinimize[{f[s], 0 ≤ s < 2Pi}, {s}, Method → "DifferentialEvolution"];
m0:=Evaluate[NMX[[1]]]; m1:=Evaluate[NMN[[1]]];
s0:=Evaluate[NMX[[2]][[1, 2]]];
s1:=Evaluate[NMN[[2]][[1, 2]]];
```

Step II. Choose p and q, set $\delta_0(r) = -m_1 r$ and $\delta_1(r) = m_0 r$ and define the periods ℓ_r. Then define the 1-parameter family of periodic functions $K(-, r)$ as explained in the general scheme and visualize the graphs of the functions:

```
p:= − 1; q:=5;
δ0[r_]:= − r ∗ m1; δ1[r_]:=r ∗ m0;
ℓ[r_]:=Evaluate[2Pi + δ1[r] + δ0[r]];
B[s_, m_, n_]:=UnitStep[s − m] ∗ (1 − UnitStep[s − n]);
f1[s_, r_]:=B[s, 0, s0] ∗ f[s];
f2[s_, r_]:=B[s, s0, δ0[r] + s0] ∗ m0;
f3[s_, r_]:=B[s, δ0[r] + s0, δ0[r] + s1] ∗ f[s − δ0[r]];
f4[s_, r_]:=B[s, δ0[r] + s1, δ0[r] + δ1[r] + s1] ∗ m1;
f5[s_, r_]:=B[s, δ0[r] + δ1[r] + s1, ℓ[r]]f[s − δ0[r] − δ1[r]];
F0[s_, r_]:=f1[s, r] + f2[s, r] + f3[s, r] + f4[s, r] + f5[s, r];
F[s_, r_]:=Sum[F0[s − j ∗ ℓ[r], r], {j, −q, q}];
K[s_, r_]:=F[s, r] + 2Pi∗p/(ℓ[r]∗q);
(*Visualization*)
VK[r_]:=ParametricPlot[{s, K[s, r]}, {s, 0, ℓ[r] − 10^(−5)},
PlotStyle → {{Thickness[0.006], Blue}},
PlotRange → All, ImageSize → {400, 200}];
GraphicsGrid[{{VK[0], VK[1]}, {VK[2], VK[2.78]}, {VK[3.2], VK[6]}},
Frame → All]
```

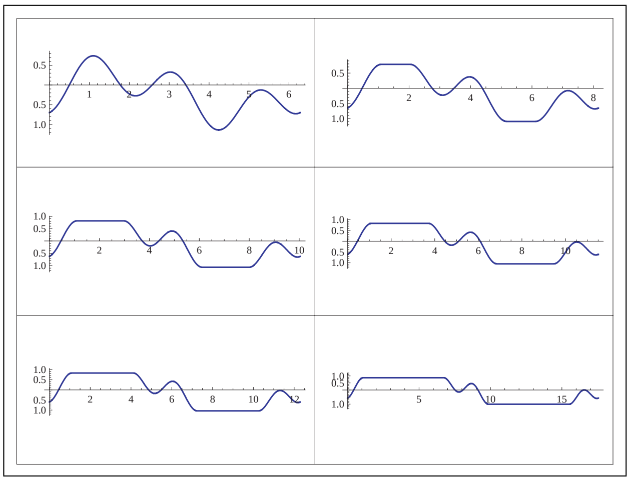

FIGURE 3. $K(-, r)$, $r = 0, 1, 2.78, 3.2, 6$.

Step III. Compute and visualize the signature:

```
MDk[S_, r_]:=D[K[S, r], S];
SG[S_, r_]:=Evaluate[{K[S, r], MDk[S, r]}];
(*Visualization*)
VisualizeSignature[r_]:=ParametricPlot[SG[s, r], {s, 0, ℓ[r]},
PlotStyle → {{Thickness[0.006], Blue}}, Axes → True, AspectRatio → 1/1.1,
PlotPoints → 100, PlotRange → All, ImageSize → {400, 400}];
VisualizeSignature[0]
```

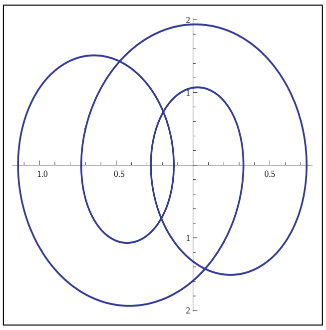

FIGURE 4. The signature of the family.

Step IV. Solve the Frenet system and visualize the curves of the family:

```
solution[1][r_]:=
NDSolve [{x'[t] == y[t], x[0] == 0, y'[t] == K[t, r] * z[t], y[0] == 1,
z'[t] == −K[t, r] * y[t], z[0] == 0}, {x, y, z}, {t, 0, 2 * q * ℓ[r]}];
solution[2][r_]:=
NDSolve [{x'[t] == y[t], x[0] == 0, y'[t] == K[t, r] * z[t], y[0] == 0,
z'[t] == −K[t, r] * y[t], z[0] == 1}, {x, y, z}, {t, 0, q * ℓ[r]}];
S[1][t_, r_]:={x[t], y[t], z[t]}/.solution[1][r];
S[2][t_, r_]:={x[t], y[t], z[t]}/.solution[2][r];
γ[t_, r_]:={S[1][t, r][[1]][[1]], S[2][t, r][[1]][[1]]};
(*Visualization*)
FG[r_]:=ParametricPlot[Evaluate[γ[t, r]], {t, 0, q * ℓ[r]},
PlotStyle → {{Thickness[0.006], Blue}}, Axes → True, AspectRatio → Automatic,
PlotPoints → 100, PlotRange → All, ImageSize → {400, 400}];
GraphicsGrid[{{FG[0], FG[1], FG[2.78]}, {FG[2.79], FG[3.2], FG[6]}}, Frame → All]
```

Remark 8. The curves γ_r are simple for $0 \le r < r_0$ and have self-intersections for $r \ge r_0$, where $r_0 \approx 2.79$. The curves have symmetry group $\mathbb{Z}_5$ and are not congruent to each other. Moreover γ_0 has only isolated vertices, while the set of vertices of γ_r, $r > 0$, has interior points. As a consequence, γ_0 and γ_r, $r > 0$, are not locally congruent to each other. Note that, $K(s, r) = c_0 > 0$ on the intervals $[s_0 + j \cdot \ell_r, s_0 + \delta_0(r) + j \cdot \ell_r]$, $j = 0, \dots, q-1$. The corresponding arcs are pieces of convex circles of radius $\|c_0\|^{-1}$. Similarly, $K(s, r) = c_1 < 0$ on the intervals $[s_0 + \delta_0(r) + j \cdot \ell_r, s_1 + \delta_0(r) + \delta_1(r) + j \cdot \ell_r]$. The corresponding arcs are pieces of concave circles of radius $\|c_1\|^{-1}$.

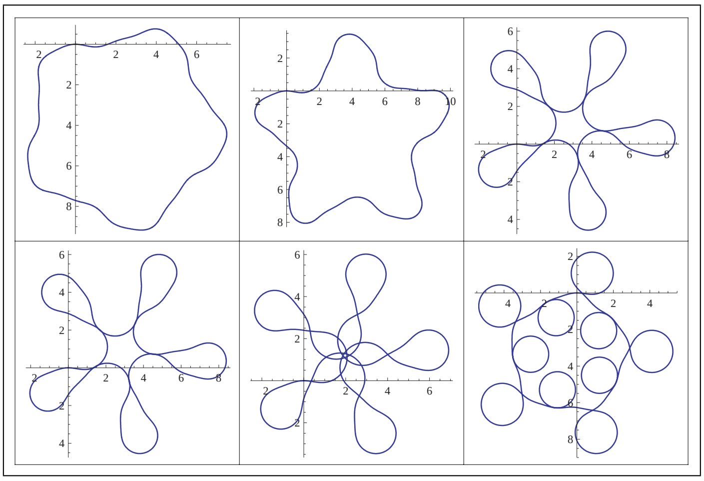

FIGURE 5. The curves γ_r, $r = 0, 1, 2.78, 3.2, 6$.

Example 2. The code can be applied to any other periodic function such that $s_0 < s_1$. For instance, take

$$f(s) = \frac{1}{2}\left(\sin(s) - \cos(3s) + \frac{1}{7}\cos(7s) - \frac{1}{11}\sin(11s)\right)$$

and set $p = 1$, $q = 3$ (i.e. a family with symmetry group $\mathbb{Z}_3$). The graphs of the curvatures $K(-, r)$, $r = 0, 1, 2, 3, 4, 5$, the signature and the curves of the family are given in Figures 6, 7 and 8, respectively.

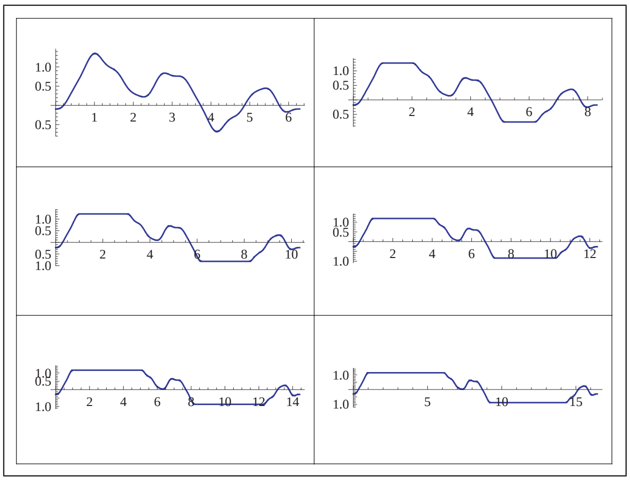

FIGURE 6. $K(-, r)$, $r = 0, 1, 2, 3, 4, 5$.

3.3.1. *Conclusions.* The previous examples show that the shape of a simple closed curve is not necessarily determined by its signature and suggest that such a phenomenon holds for any non-convex simple closed curve with a non-trivial symmetry

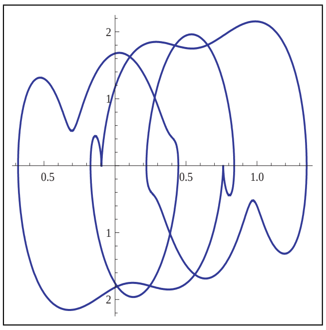

FIGURE 7. The signature of the family.

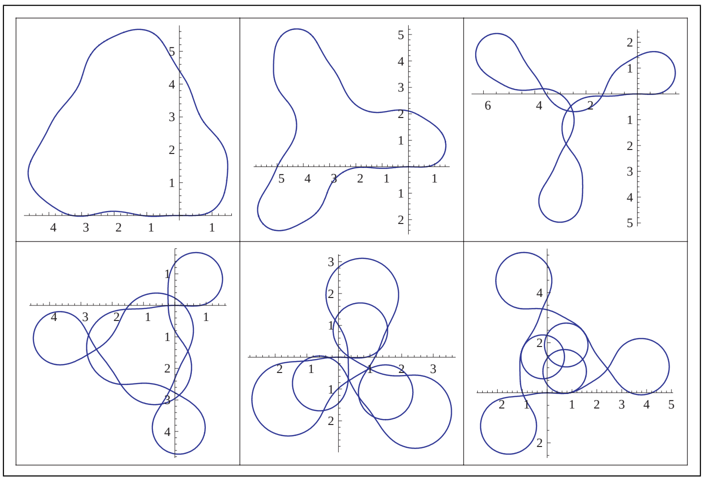

FIGURE 8. The curves γ_r, $r = 0, 1, 2, 3, 4, 5$.

group. This is in contrast with the claim of Theorem 2.3 in reference [6]. The argument used in [6] is based on the implicit function theorem, i.e. on the assumption that the signature can be locally parametrized as the graph of a function. The argument fails exactly at the points of the signature which correspond to the vertices of the curve, i.e. at the intersections of the signature with the x-axis. In the next section, we will show that the knowledge of the symmetry group and of the basic topological and metrical invariants such as length and turning number is not enough to determine the shape of a simple closed curve by its signature.

4. ISOSIGNED DEFORMATIONS: PROOF OF THEOREM 2

We now turn to the proof of Theorem 2. We shall construct families of isosigned smooth strictly convex simple closed curves which are not congruent to each other.

Proof of Theorem 2. Let $a > 0$ such that $\eta := \pi - 2a > 0$, and consider the smooth non-negative function

$$f(s) = \begin{cases} 0 & \text{if } \quad s \leq -a/2, \\ \exp\left(\dfrac{1}{s} - \dfrac{1}{s+a/2}\right) & \text{if } \quad -a/2 < s < 0, \\ 0 & \text{if } \quad s \geq 0. \end{cases}$$

Then the function

$$h(s) = \frac{\int_{-a/2}^{s} f(u)du}{\int_{-\infty}^{\infty} f(u)du} \tag{4.1}$$

is non-negative, and takes value 1 for $s \geq 0$ and value 0 for $s \leq -a/2$. Next, for each $r \in [0, \eta)$, consider the "plateau" function

$$h_r(s) = h\left(s + \frac{r}{2}\right)\left[1 - h\left(s - \frac{r}{2}\right)\right].$$

Take $\epsilon \in (0, 1)$ and, for each $r \in [0, \eta)$, let κ_r be the strictly positive periodic function of minimal period 2π given by

$$\kappa_r(s) = \frac{1}{q} + \frac{\epsilon}{q}\left[h_r\left(s - \frac{\pi}{2}\right) - h_r\left(s - \frac{3\pi}{2}\right)\right], \quad s \in [0, 2\pi).$$

Let $\gamma_r : \mathbb{R} \to \mathbb{R}^2$ be the unit-speed parameterized curve with curvature κ_r and

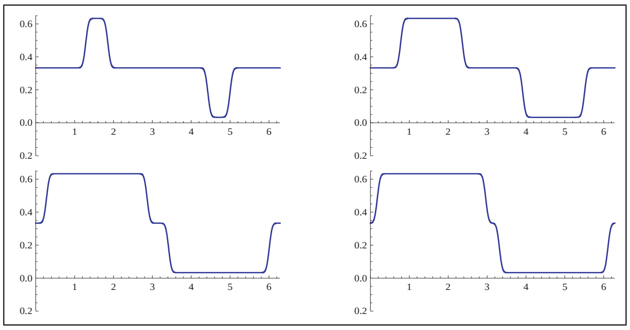

FIGURE 9. The functions κ_r for $q = 3$, $\epsilon = 0.9$ and $r = 0, 0.5, 1, 1.1$.

initial conditions

$$\gamma_r(0) = (0, 0), \quad \dot{\gamma}_r(0) = (1, 0).$$

By construction we have

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_r(u)du = \frac{1}{q}.$$

From Corollary 5, it follows that γ_r parametrizes a smooth strictly convex simple closed curve Γ_r of length $2q\pi$. Since the functions $\{\kappa_r\}_{r\in[0,\eta)}$ are first-order deformations of each other (cf. Remark 6), the curves Γ_r, $r \in [0, \eta)$, all have the same signature. On the other hand, for different values of r, the functions κ_r cannot be obtained from each other by just a reparametrization of the form $s \to s + c$. Thus, for $r \neq r'$, the curves Γ_r and $\Gamma_{r'}$ are not congruent to each other. Moreover, Γ_0 has isolated vertices, while the curves Γ_r, $r \in (0, \eta)$, have only non-isolated vertices.

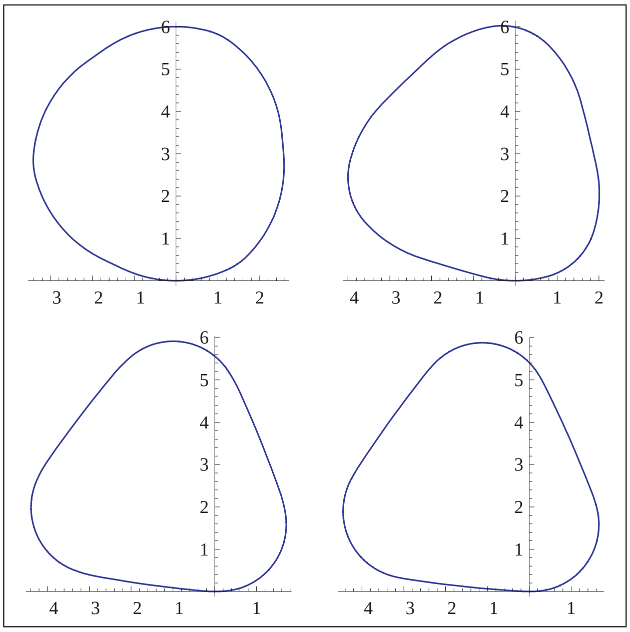

FIGURE 10. The curves with curvatures κ_r, for $q = 3$, $\epsilon = 0.9$ and $r = 0, 0.5, 1, 1.1$. The curves have length $L = 2q\pi$.

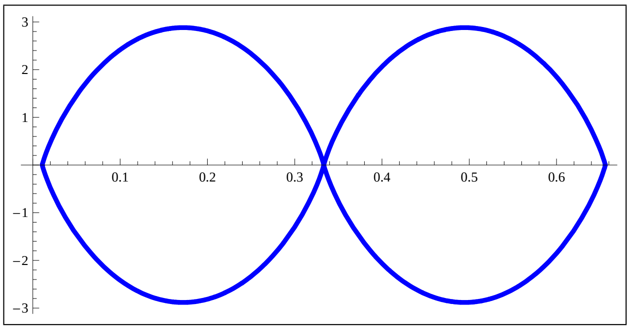

FIGURE 11. The signature of the family.

This implies that Γ_0 and Γ_r cannot be locally congruent, which concludes the proof of the theorem. $\square$

Example 3. Here we give the source codes related to Figures 9, 10 and 11. For the purpose of numerical computations we replace the function h in (4.1) with a suitably normalized "error function".

Step 1. Define the curvature functions κ_r:

```
q:=3; a:=0; b:=2 * q * Pi;
h[t_, r_]:=(1/2)(Erf[(9.5/0.8)(t + r + 0.8) − 6] + 1);
```

```
B[t_, r_, c_]:=h[-t + c, r]h[t - c, r];
Bump[t_, r_]:=B[t, r, Pi/2] - B[t, r, 3Pi/2];
k[t_, r_]:=(1/q) + (0.9/(q))Sum[(Bump[t - 2j * Pi, r]), {j, 0, q}];
```

Step 2. Compute the curves γ_r and the signature of the family:

```
solution[1][r_]:=NDSolve [{x'[t] == y[t], x[0] == 0, y'[t] == k[t, r] * z[t], y[0] == 1,
z'[t] == -k[t, r] * y[t], z[0] == 0}, {x, y, z}, {t, a, b}];
solution[2][r_]:=NDSolve [{x'[t] == y[t], x[0] == 0, y'[t] == k[t, r] * z[t], y[0] == 0,
z'[t] == -k[t, r] * y[t], z[0] == 1}, {x, y, z}, {t, a, b}];
S[1][t_, r_]:={x[t], y[t], z[t]}/.solution[1][r];
S[2][t_, r_]:={x[t], y[t], z[t]}/.solution[2][r];
γ[t_, r_]:={S[1][t, r][[1]][[1]], S[2][t, r][[1]][[1]]};
dk[t_, r_]:=Evaluate[D[k[t, r], t]];
PP1[t_, r_]:=Evaluate[{k[t, r], dk[t, r]}];
```

5. Symmetries and signatures: Proof of Theorem 3

In this section, we show the impossibility of detecting the symmetries of a simple closed curve from the index of its signature map. The proof is based on the study of a special family of smooth simple closed curves arising as boundaries of star-shaped type domains.

5.1. **Cogwheels.** Let n be a positive integer and consider the closed intervals

$$I_j = \left[\frac{2(j-1)\pi}{n}, \frac{2j\pi}{n}\right]$$

so that $[0, 2\pi] = \bigcup_{j=1}^{n} I_j$. For each $j \in \{1, \dots, n\}$ consider a smooth function

$$r_j : \mathbb{R} \to \mathbb{R}^+, \quad \mathrm{supp}(r_j) \subset \dot{I}_j$$

and assume that among the r_j at least one is non constant. Let r_0 be a positive constant and denote by $\rho : \mathbb{R} \to \mathbb{R}^+$ the unique periodic extension, with period 2π, of the function

$$r_0 + \sum_{j=1}^{n} r_j : [0, 2\pi] \to \dot{\mathbb{R}}^+.$$

Such a ρ will be referred to as an *n-bump radial function.* Correspondingly, consider the smooth simple closed curve Γ, parametrized by

$$\gamma : t \in \mathbb{R} \mapsto \rho(t)(\cos t, \sin t) \in \mathbb{R}^2.$$

Definition 3. The curve Γ is called an *n-cogwheel* with radial function ρ, inner radius r_0, and ordered bumps $(r_1, \dots, r_n)$.

The velocity and the curvature of the parametrization γ are given by

$$\begin{cases} v(t) = \sqrt{\rho^2 + \rho'^2}, \\ \kappa_\gamma(t) = \dfrac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho^2 + \rho'^2)^{\frac{3}{2}}}. \end{cases}$$

It follows that the signature of γ is parametrized by signature map

$$\sigma_\gamma : t \mapsto \sigma_\gamma(t) = \left(\kappa(t), \frac{1}{v(t)}\kappa'(t)\right)$$

and that the corresponding index is

$$\operatorname{ind}\sigma_\gamma = \min\left\{\#\sigma_\gamma^{-1}(\zeta) \,|\, \zeta \in \mathcal{S}(\gamma),\ \sigma_\gamma^{-1}(\zeta) \in [0, 2\pi)\right\}.$$

5.2. **Combinatorics of cogwheels.** Let $\mu \in S_n$ be a permutation on $\{1, \ldots, n\}$. For each j, $j = 1, \ldots, n$, define the translation

$$\tau_{\mu(j)}(t) = t + \frac{2(j - \mu(j))}{n}\pi$$

so that $\tau_{\mu(j)}(I_{\mu(j)}) = I_j$ and set $\mu \cdot r_j = r_j \circ \tau_{\mu(j)}$. Let $\mu \cdot \rho$ be the periodic extension of

$$r_0 + \sum_{j=1}^{n} \mu \cdot r_j$$

and denote by $\mu \cdot \gamma$ the corresponding n-cogwheel.

Remark 9. The cogwheel $\mu\cdot\gamma$ is obtained from γ by permuting its cogs. This defines an action of the symmetric group S_n on the set of n-cogwheels. All the curves $\mu \cdot \gamma$, for $\mu \in S_n$, are locally congruent to each other,[4] but for a generic choice of the n-bump radial function ρ they are not globally congruent. This happens, for instance, if $\max r_i \neq \max r_j$, $i \neq j$, $i, j = 1, \ldots, n$. In particular, this also implies that the symmetry groups of cogwheels are not preserved under the action of the symmetric group.

Lemma 6. *The signature and the index of n-cogwheels are invariant under the action of the symmetric group S_n.*

Proof. From $\mu \cdot \rho_{|I_{\mu(j)}} = \rho \circ \tau_{\mu(j)}$, we have

$$\begin{aligned} \mu \cdot v(t)_{|I_{\mu(j)}} &= v \circ \tau_{\mu(j)} \\ \mu \cdot \kappa_{|I_{\mu(j)}} &= \kappa \circ \tau_{\mu(j)}. \end{aligned}$$

Therefore,

$$\mu \cdot \sigma_{|I_{\mu(j)}} = \sigma \circ \tau_{\mu(j)}, \quad j = 1, \ldots, n, \tag{5.1}$$

which implies $\mu \cdot \mathcal{S} \subset \mathcal{S}$. Similarly, by interchanging the role of $\mu \cdot \rho$ and ρ, we have that $\mathcal{S} \subset \mu \cdot \mathcal{S}$, and the first assertion is proved. It also follows from (5.1) that the indices of the signature maps coincide. The proof goes as follows. For each j, let

$$\tilde{I}_j = \left[\frac{2(j-1)\pi}{n}, \frac{2j\pi}{n}\right)$$

so that $[0, 2\pi)$ is the disjoint union of the $\tilde{I}_j$. Consider the bijection

$$\beta : [0, 2\pi) \to [0, 2\pi)$$

defined by $\beta_{|\tilde{I}_{\mu(i)}} = \tau_{\mu(i)}{}_{|\tilde{I}_{\mu(i)}}$. Then β induces a bijection between $(\mu \cdot \sigma)^{-1}(\zeta) \cap [0, 2\pi)$ and $\sigma^{-1}(\zeta) \cap [0, 2\pi)$, which proves the required result. □

Example 4 (Combinatorics of a cogwheel)**.** Consider the 3-bump radial function given in Figure 12. The combinatorics of this 3-bump radial function determines the six configurations given in Figure 13. The corresponding six 3-cogwheels and their curvatures are represented in Figures 14 and 15, respectively. The signature of the six 3-cogwheels is made of three pieces (one for each bump). Since the bumps

[4] This follows from the fact that ρ and $\mu \cdot \rho$ are locally congruent, i.e., $\forall t_0 \in \mathbb{R}$ there exists $\epsilon > 0$ and $t_1 \in \mathbb{R}$ such that $\rho_{|(t_0-\epsilon, t_0+\epsilon)}(t) = (\mu \cdot \rho)(t + t_1)$, and conversely.

are different from each other, the three pieces are also different and the index of the signature is 1 (cf. Figure 16). The source codes are given below.

Step 1. Define the 3-bump radial function and its combinatorics. The numerical values are $\delta = 0.5$, $n = 3$. The integer $p = 1, \dots, 6$ labels the six possible configurations:

```
g[t_]:=(1/2)(Erf[t − 6] + 1);
h[t_, a_]:=g[(9.5/a)t];
n[t_, a_, r_]:=h[t + a + r, a];
Bump[t_, r_, a_, c_]:=n[−t + c, a, r]n[t − c, a, r];
L[j_, n_, p_]:=Permutations[Table[m, {m, 1, n}]][[p]][[j]];
r[t_, δ_, n_, p_]:=2.5 + Sum[(0.3 ∗ Cos[(n)(L[j, n, p] + 1) ∗ t])
Bump[t, (1 − δ)Pi/n, δ(Pi/n), (2Pi/n) ∗ (j − 1) + Pi/n], {j, 1, n}];
```

Step 2. Define the cogwheels, their curvatures and the signature:

```
γ[t_, δ_, n_, p_]:=r[t, δ, n, p]{Cos[t], Sin[t]};
v[t_, δ_, q_, p_]:=Sqrt[D[γ[t, δ, q, p], t].D[γ[t, δ, q, p], t]];
k[t_, δ_, n_, p_]:=
v[t, δ, n, p]^−3∗(D[γ[t, δ, n, p][[1]], t] ∗ D[D[γ[t, δ, n, p][[2]], t], t]−
D[γ[t, δ, n, p][[2]], t] ∗ D[D[γ[t, δ, n, p][[1]], t], t]);
dk[t_, δ_, n_, p_]:=v[t, δ, n, p]^−1 D[k[t, δ, n, p], t];
Sgn[t_, δ_, n_, p_]:={k[t, δ, n, p], dk[t, δ, n, p]};
```

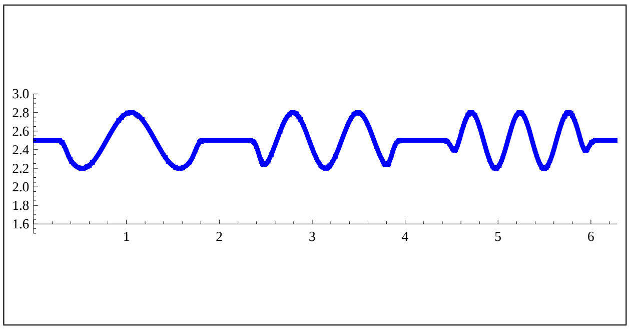

Figure 12. A 3-bump radial function.

5.3. **Proof of Theorem 3.** Let G_1, G_2 be two finite subgroups of SO(2) of order $q_1 \geq 1$ and $q_2 \geq 1$, respectively. Set $n = 2q$, where $q = q_1 q_2$, and decompose $[0, 2\pi]$ into the union of the n intervals I_{2k-1}, I_{2k}, $k = 1, \dots, q$. Next, consider two smooth non-constant functions

$$r_1, r_2 : \mathbb{R} \to \mathbb{R}^+, \quad \mathrm{supp}(r_1) \subset \dot{I}_1, \, \mathrm{supp}(r_2) \subset \dot{I}_2$$

so that

$$\max r_1 \neq \max r_2.$$

Now, define the functions

$$r_{2k-1} := r_1 \circ \tau_{2k-1}, \quad r_{2k} := r_2 \circ \tau_{2k}, \quad k = 1, \dots, q,$$

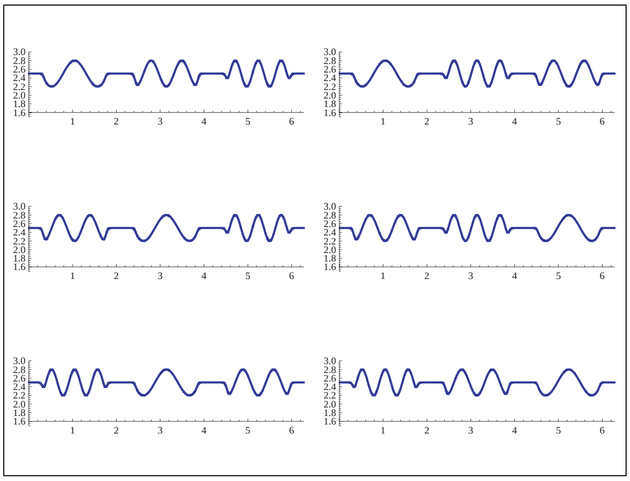

FIGURE 13. The six configurations of the combinatorics.

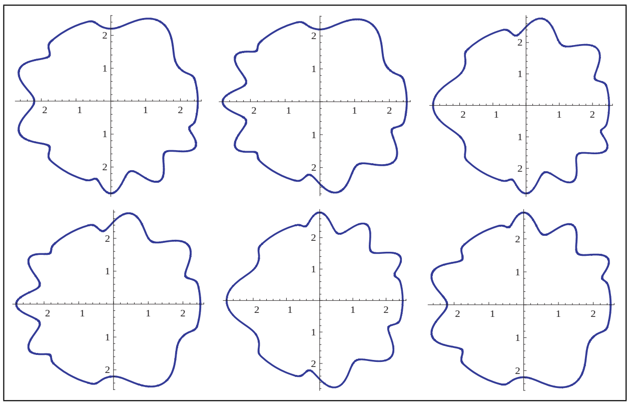

FIGURE 14. The six 3-cogwheels.

where

$$\begin{cases} \tau_{2k-1} : t \in I_{2k-1} \mapsto t - 2(k-1)\pi/q \in I_1, \\ \tau_{2k} : t \in I_{2k} \mapsto t - 2(k-1)\pi/q \in I_2, \end{cases} \qquad k = 1, \ldots, q,$$

and consider the "fully symmetric" bump function ρ, of period $2\pi/q$, given by the periodic extension of

$$r_0 + \sum_{j=1}^{n} r_j,$$

where r_0 is any assigned non-null constant.

Next, let $\mu_1, \mu_2 \in S_n$ be the permutations defined by

$$\begin{cases} \mu_1(2\ell - 1) = 2\ell,\ \mu_1(2\ell) = 2\ell - 1, & \ell = q_1, 2q_1, \ldots, q \\ \mu_1(\ell) = k, & \ell \neq q_1, 2q_1, \ldots, q \end{cases}$$

$$\begin{cases} \mu_2(2\ell - 1) = 2\ell,\ \mu_2(2\ell) = 2\ell - 1, & \ell = q_2, 2q_2, \ldots, q \\ \mu_2(\ell) = k, & \ell \neq q_2, 2q_2, \ldots, q \end{cases}$$

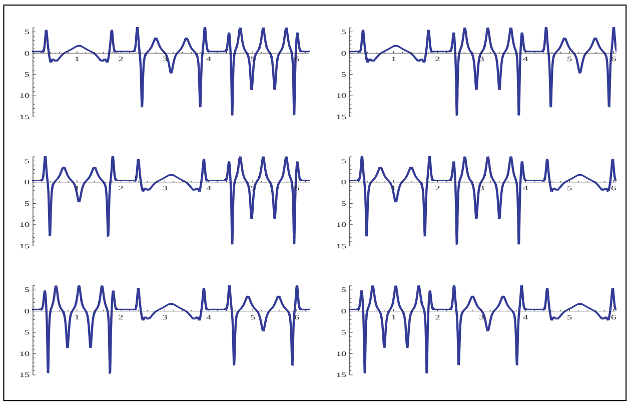

FIGURE 15. The curvature functions of the six 3-cogwheels.

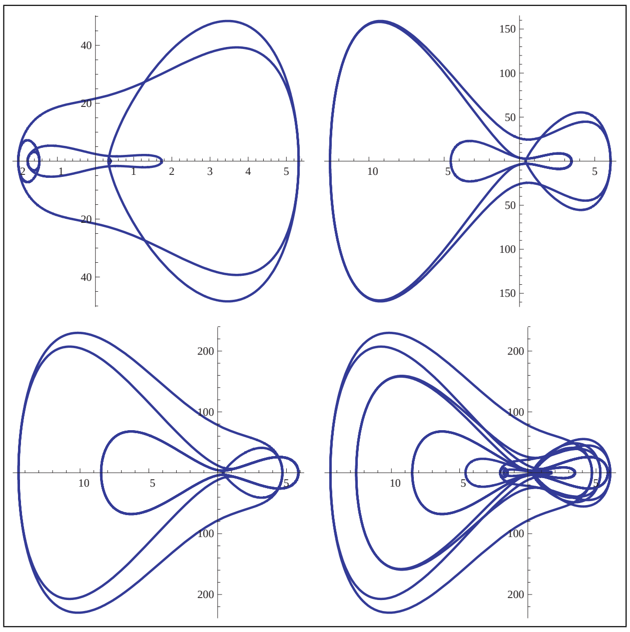

FIGURE 16. The fundamental pieces of the signature and the signature.

The bump functions $\mu_1 \cdot \rho$ and $\mu_2 \cdot \rho$ have period $2\pi/q_2$ and $2\pi/q_1$, respectively. This implies that the cogwheel with radial function $\mu_1 \cdot \rho$ has symmetry group isomorphic to $\mathbb{Z}_{q_2}$, while the symmetry group of the cogwheel corresponding to $\mu_2 \cdot \rho$ is isomorphic to $\mathbb{Z}_{q_1}$. This, together with Lemma 6, completes the proof of Theorem 3.

Example 5 (Cogwheels with same signatures and indices)**.** Consider the “fully symmetric” 12-bump radial function of period $\pi/3$ given in Figure 17. The four configurations of this 12-bump function are represented in Figure 18. The first has period 2π, the second has period π, the third has period $3\pi/3$, and the last one

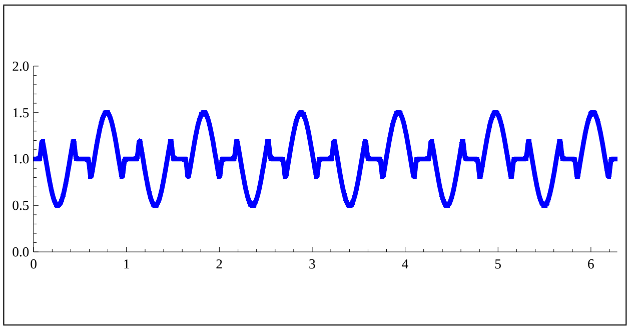

Figure 17. The 12-bump radial function.

has period $\pi/3$. The four corresponding cogwheels are illustrated in Figure 19,

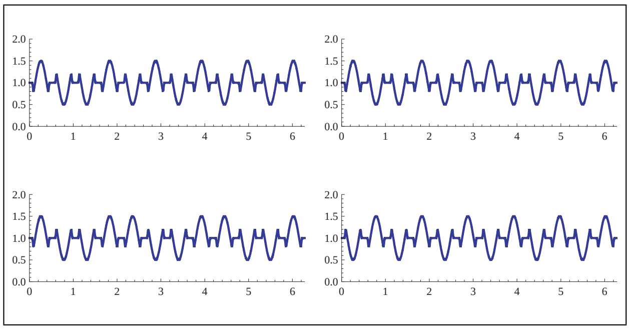

Figure 18. The 4 configurations of the 12-bump radial function.

while the respective curvature functions are given in Figure 20. The first cogwheel curve has trivial symmetry group, the second has symmetry group $\mathbb{Z}_2$, the third has symmetry group $\mathbb{Z}_3$, while the last one has symmetry group $\mathbb{Z}_6$. The signature of these four cogwheels has index 6 and is given in Figure 21.

References

[1] M. Boutin, Numerically invariant signature curves, *Int. J. Computer Vision* **40** (2000), 235–248.

[2] R. C. Brower, D. A. Kessler, J. Koplik, H. Levine, Geometrical models of interface evolution, *Physical Review A* **29** (1984), 1335–1342.

[3] A. M. Bruckstein, D. Shaked, Skew symmetry detection via invariant signature, CIS Report No. 9419, Technion, IIT, Haifa, December 1994.

[4] A. M. Bruckstein, A. N. Netravali, On differential invariants of planar curves and recognizing partially occluded planar shapes, *Ann. Math. Artificial Intelligence* **13** (1995), 227–250.

[5] A. M. Bruckstein, N. Katzir, M. Lindenbaum, M. Porat, Similarity invariant signatures for partially occluded planar shapes, *Int. J. Computer Vision* **7** (1992), 271–285.

[6] E. Calabi, P. J. Olver, C. Shakiban, A. Tannenbaum, S. Haker, Differential and numerically invariant signature curves applied to object recognition, *Int. J. Computer Vision* **26** (1998), 107–135.

[7] K.-S. Chou, C. Qu, Integrable equations arising from motions of plane curves, *Phys. D* **163** (2002), 9–33.

[8] M. Fels, P. J. Olver, Moving coframes, II. Regularization and theoretical foundations, *Acta Appl. Math.* **55** (1999), 127–208.

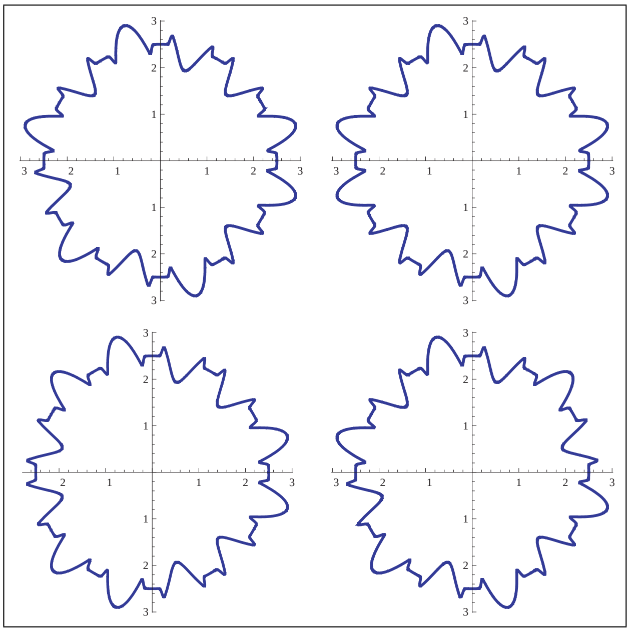

FIGURE 19. The four cogwheels.

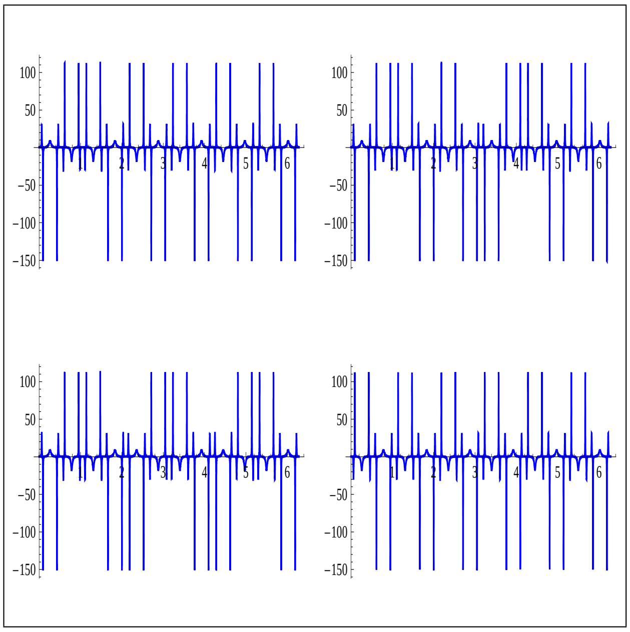

FIGURE 20. The curvatures of the four cogwheels.

[9] A. Ferrández, A. Giménez, P. Lucas, Geometrical particle models on 3D null curves, *Phys. Lett. B* **543** (2002), 311–317; hep-th/0205284.

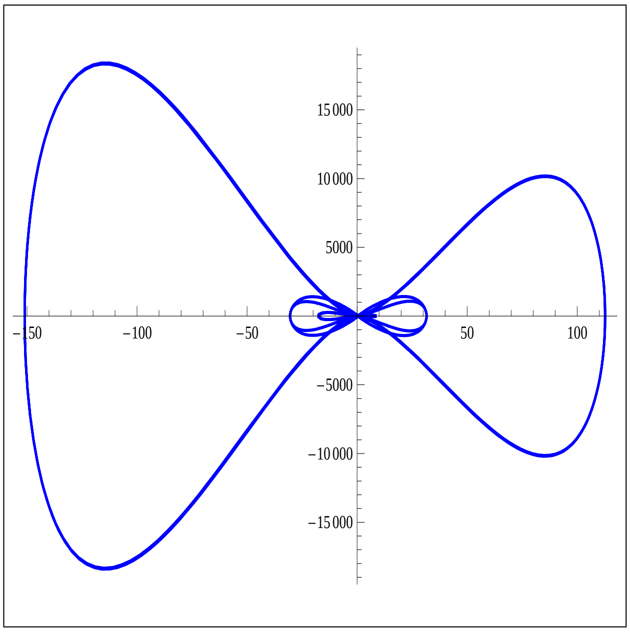

FIGURE 21. The signature of the four cogwheels.

[10] H. Gluck, The converse to the four vertex theorem, *Enseignement Math. (2)* **17** (1971), 295–309.

[11] J. D. E. Grant, E. Musso, Coisotropic variational problems, *J. Geom. Phys.* **50** (2004), 303–338; math.DG/0307216.

[12] P. A. Griffiths, *Exterior differential systems and the calculus of variations*, Progr. Math., 25, Birkhäuser, Boston, 1982.

[13] S. Manay, B.-W. Hong, A. J. Yezzi, S. Soatto, Integral Invariant Signatures, 87–99, Lecture Notes in Computer Science, Volume 3024, Springer, Berlin, Heidelberg, 2004.

[14] E. Musso, L. Nicolodi, Reduction for the projective arclength functional, *Forum Math.* **17** (2005), 569-590.

[15] E. Musso, L. Nicolodi, Closed trajectories of a particle model on null curves in anti-de Sitter 3-space, *Classical Quantum Gravity* **24** (2007), no. 22, 5401–5411.

[16] E. Musso, L. Nicolodi, Reduction for constrained variational problems on 3-dimensional null curves, *SIAM J. Control Optim.* **47** (2008), no. 3, 1399-1414.

[17] P. J. Olver, Moving frames–in geometry, algebra, computer vision, and numerical analysis, *Foundations of computational mathematics* (Oxford, 1999), 267–297, London Math. Soc. Lecture Note Ser., 284, Cambridge Univ. Press, Cambridge, 2001.

[18] P. J. Olver, Moving frames, *Journal of Symbolic Computation* **36** (2003), 501–512.

[19] P. J. Olver, Invariant signatures, Breckenridge, March 2007; Seminars and Conference Talks at `http://www.math.umn.edu/~olver`.

(E. MUSSO) DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI 24, I-10129 TORINO, ITALY

E-mail address: `emilio.musso@polito.it`

(L. NICOLODI) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI PARMA, VIALE G. P. USBERTI 53/A, I-43100 PARMA, ITALY

E-mail address: `lorenzo.nicolodi@unipr.it`